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Backward Stochastic Differential Equations and Stochastic Optimal Control in Finance

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Abstract

This thesis presents the Backward Stochastic Differential Equations (BSDE), the motivation of this theory and the first theorems of existence and uniqueness. Also, we show the principal results for one dimensional linear BSDEs and some examples using BSDE. We give an introduction to the quadratic case based on the article written by Kobylanski. We implement some numerical methods using MATLAB for solving BSDE with quadratic growth, such as the Euler-Maruyana approximation and the truncation method. We tackle the stochastic optimal control theory. We give the intuition of the dynamic programming principle, a formal deduction of the Hamilton-Jacobi-Bellman equation and the relationship with the BSDE. Finally, all these topics are combined in an example in finance with numerical results.

Dedicado a la memoria de mis hermanos, Erika (1975-2008) y Julián (1973-2010), de quienes aprendí a vivir cada día de mi vida al máxmo y a luchar por hacer de mis sueños una realidad.

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Contents

Introduction

Backward Stochastic Differential Equations (BSDEs) are stochastic differential equations with terminal value. The theory of BSDEs has found wide applications in areas such as stochastic optimal control, theoretical economics and mathematical finance problems.

This thesis starts by presenting Backward Stochastic Differential Equations, the motivation of this theory [2] and the first theorems of existence and uniqueness of Pardoux and Peng [11], also we show the principal results for one dimensional linear BSDEs and some examples using BSDE [4].

Then we give an introduction to the quadratic case based on the article written by Kobylanski [8]. We give sufficient conditions for the existence of at least one solution of a BSDE in the quadratic case and a theorem of the stability of the solution. This theorem gives us an idea of dimensions of the solution and then we can make numerical approximations to solution.

To deal with BSDE with quadratic growth we consider the truncation method. This method consists of defining a sequence of generators that satisfy the Lipschitz condition. This sequence has the property that the limit is the generator with quadratic growth. Then we show Euler-Maruyana approximation. This is a numerical method that approximates numerical solution of a stochastic differential equation. It is used because it is the simplest numerical method and does not require much prior knowledge [9], [13].

We tackle the stochastic optimal control theory, give the intuition of the dynamic programming principle and a formal deduction of the Hamilton-Jacobi-Bellman equation. We show the relationship between the stochastic optimal control theory and the BSDE [5], [12],[15]. Finally, all this theory is intertwined in an example in finance which concludes with numerical results.

In Section 3.4.1 we introduce the financial model [10], in Section 3.4.2 we show the properties of the generator of the BSDE with nonlinearities of quadratic type. In Section 3.4.3 we solve the Optimal Control via BSDE based on the results of [15] concerning the solution of the problem of exponential expected utility maximization in terms of stochastic control problems and BSDE. Finally, in Section 3.4.4 we solve the stochastic control problem via BSDE using the results of Chapter 2.4.

Chapter 1

Backward Stochastic Differential Equations

1.1 Background

In 1671 Newton worked on the theory of "Fluxions". His research was related to "fluxional equations" what we would now call differential equations. The mathematician and philosopher Gottfried Wilhelm Leibniz also worked on differential equations and found a method to solve linear differential equations of first order. In 1690, Jakob Bernoulli showed that the problem of determining the isochrone is equivalent to solving a nonlinear differential equation of first order by the method of separable variables.

In the history of Differential Equations the principal character was Leonard Euler. He introduced several methods for low-order equations, the concept of integrating factor, the theory of linear equations of arbitrary order, the application development method of power series among other things. The problems of existence and uniqueness of the solution became important thanks to Niels Henrik Abel and Augustin-Louis Cauchy.

The earliest work on stochastic differential equations was to describe brownian motion in Einstein's work in 1956 and simultaneously by Smoluchowski. However, one of the first work on the brownian motion is attributed to Bachelier in the "Theory of Speculation" in 1900. This work was followed by Langevin. Later, Itô and Stratonovich began to stochastic differential equations on more solid mathematical basis.

Finally, BSDEs were introduced by Bismut in 1973 for the linear case and were subsequently taken up and generalized by Padoux and Peng in 1990.

1.2 Notation and definitions

Throughout this thesis, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space on which is defined a n-dimensional standard brownian motion W_t, such that $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration of W_t, augmented by all the $\mathbb{P}-$nulls sets.

We will use the following spaces of random variables or processes:

- For $x \in \mathbb{R}^{n\times m}$, $y \in \mathbb{R}^{k\times n}$,$|x| := \sqrt{\text{trace}(xx')}$ denotes the euclidean norm, while the inner product is given by $(x, y) := \text{trace}(xy')$, where x' denotes the transpose of x.
- $L^2_T\left(\mathbb{R}^d\right)$ is the space of $\mathcal{F}_T$-measurable random variables $X : \Omega \to \mathbb{R}^d$, such that $\|X\|^2 := \mathbb{E}\left(|X|^2\right) < \infty$.
- $H^2_T\left(\mathbb{R}^d\right)$ is the space of all predictable processes $\phi : \Omega \times [0, T] \to \mathbb{R}^d$ such that $\|\phi\|^2 := \mathbb{E}\int_0^T |\phi_t|^2\, dt < \infty$, i.e., the set of square integrable processes.
- $H^1_T\left(\mathbb{R}^d\right)$ is the space of all predictable processes $\phi : \Omega \times [0, T] \to \mathbb{R}^d$ such that $\mathbb{E}\sqrt{\int_0^T |\phi_t|^2\, dt} < \infty$.
- Sometimes the following notations will be used for simplification: $L^2_T\left(\mathbb{R}^d\right) = L^2_T$, $H^1_T\left(\mathbb{R}^d\right) = H^1_T$, and $H^2_T\left(\mathbb{R}^d\right) = H^2_T$.

Consider the BSDE

$$-\,dY_t = f\left(t, Y_t, Z_t\right) dt - Z'_t dW_t, \quad Y_T = \xi, \tag{1.1}$$

or equivalently

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s\right) ds - \int_t^T Z'_s dW_s, \tag{1.2}$$

where:

- The terminal value is a $\mathcal{F}_T$-measurable random variable $\xi : \Omega \to \mathbb{R}^d$.

- $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \to \mathbb{R}^d$ is the generator and is $\mathcal{P}\otimes\mathcal{B}^d\otimes\mathcal{B}^{n\times d}$-measurable, where $\mathcal{P}$ is the predectible $\sigma-$algebra over $\Omega \times \mathbb{R}^+$.

Definition 1.1. A solution of (1.1) is a pair (Y, Z), such that $\{Y_t : t \in [0, T]\}$ is a continuous adapted process $\mathbb{R}^d$-valued and $\{Z_t : t \in [0, T]\}$ is a predictable process $\mathbb{R}^{n\times d}$-valued such that $\int_0^T |Z_s|^2 \, ds < \infty$.

Assumption 1. *Suppose that* $f(\cdot, 0, 0) \in H_T^2$ *and* f *is uniformly Lipschitz, i.e., there exists* $C > 0$ *such that*

$$\begin{aligned} |f(\omega, t, y_1, z_1) - f(\omega, t, y_2, z_2)| \leq & C\left(|y_1 - y_2| + |z_1 - z_2|\right) \\ \forall (y_1, z_1),\ (y_2, z_2) \quad & d\mathbb{P}\otimes dt \quad c.s. \end{aligned}$$

Then the generator f *is said to be standard. Moreover, if* $\xi \in L_T^2$*, the data* (f, ξ) *is said to be standard data of a BSDE.*

1.3 Motivation

Consider the following ordinary differential equation (ODE)

$$dY(t) = 0 \quad t \in [0, T], \tag{1.3}$$

where the terminal time $T > 0$ is given. For each $\xi \in \mathbb{R}$ suppose that $Y(0) = \xi$ or $Y(T) = \xi$, such that (1.3) has a unique solution $Y(t) \equiv \xi$. On the other hand, if we consider (1.3) as a Stochastic Differential Equation SDE things are a little different. First consider $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ a filtered complete probability space, such that it defines a brownian motion W_t with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. First note that the solution (1.3) should be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ therefore specify $Y(T)$ or $Y(0)$ makes a big difference in the solution.

Consider the ODE with the following terminal condition:

$$\begin{cases} dY(t) = 0 \quad t \in [0, T] \\ Y(T) = \xi, \end{cases} \tag{1.4}$$

where $\xi \in L_T^2$, i.e. ξ is a random variable $\mathcal{F}_T$-measurable and square integrable.

Equation (1.4) is an ODE with a unique solution given by $Y(t) \equiv \xi$, which is not necessarily adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, unless ξ is constant. Note that in the case that is given an initial condition, the solution is adaptable. But in general (1.4) has no solution.

One way to deal with this problem is to reformulate (1.4) in such a way that we can ensure the adaptability of the solution to $\{\mathcal{F}_t\}_{t\geq 0}$. One way to do this is defining

$$Y(t) := \mathbb{E}\left[\xi \middle| \mathcal{F}_t\right]; \tag{1.5}$$

note that $Y(T) = \xi$, since ξ is $\mathcal{F}_T$-adapted. The Martingale Representation Theorem guarantees the existence of a stochastic process $Z \in L^2_T$ such that

$$Y(t) = Y(0) + \int_0^t Z(s)dW(s), \quad \forall t \in [0,T]. \tag{1.6}$$

From (1.5) and (1.6) we have

$$\begin{cases} dY(t) = Z(t)dW(t) \quad t \in [0,T] \\ Y(T) = \xi. \end{cases} \tag{1.7}$$

It has been reformulated (1.4) in (1.7) and more importantly, instead of seeking only one stochastic process Y which is $\{\mathcal{F}_t\}_{t\geq 0}$ -adapted as a solution to the SDE, we are seeking two processes (Y, Z). This allows the solution to be $\{\mathcal{F}_t\}_{t\geq 0}$ -adapted.

As is classical in SDE, (1.7) can be written on integral form, which can be deduced as follows. It is evaluated (1.6) with the terminal condition $Y(T) = \xi$ and solving for $Y(0)$.

$$Y(0) = \xi - \int_0^T Z(s)dW(s), \tag{1.8}$$

(1.8) is replaced in (1.6) and we obtain

$$Y(t) = \xi - \int_0^T Z(s)dW(s) + \int_0^t Z(s)dW(s) \quad \forall t \in [0,T] \text{ a.s.}, \tag{1.9}$$

simplifying,

$$Y(t) = \xi - \int_t^T Z(s)dW(s) \quad \forall t \in [0,T] \text{ a.s.} \tag{1.10}$$

This thesis will not distinguish between (1.7) and (1.10), both of which will be called backward stochastic differential equations (BSDE). It is worth mentioning that the Stochastic Differential Equation with initial condition will be called Forward Stochastic Differential Equation FSDE.

1.4 Existence and uniqueness

There are several theorems that tell us under which conditions the solution exists and is unique. The first theorem of this type was by Bismut (1979)[2], who considered the case in which the generator was linear, the following important result in existence and uniqueness theorems was given by Pardoux and Peng (1990) [11], where they gave a generalization of Bismut's idea, for the case in which the generator satisfies the Lipschitz condition. In this section we examine critically and in detail the existence and uniqueness theorems of Pardoux and Peng, and their proofs.

In Lemma 1.1 proves the existence and uniqueness for the case where the generator does not depend on (X, Y), and this is done using the martingale representation Theorem A.3 so that we do not need the Lipschitz condition for the generator f.

Proposition 1.1 is considered a generator which depends on Z, so called because of the Lipschitz condition which is necessary to prove the uniqueness using the Gronwall's Lemma A.3. For the existence we use Picard's construction that is well defined thanks to Lemma 1.1, again using the Lipschitz condition in order to prove that this construction converges to the solution.

Finally, Theorem 1.1, which is the most general, considers the generator f as a function that depends on (Y, Z) and satisfies the Lipschitz condition for (Y, Z), the idea to prove this theorem is the same as in Proposition 1.1.

Lemma 1.1. *Given $\xi \in L_T^2$ and $f \in H_T^2$, where $f : \mathbb{R}^+ \to \mathbb{R}^d$ is the generator, there exists a unique pair $(Y, Z) \in H_T^2\left(\mathbb{R}^d\right) \times H_T^2\left(\mathbb{R}^{n\times d}\right)$ such that satisfies*

$$Y_t = \xi + \int_t^T f(s)\, ds - \int_t^T Z_s' dW_s. \tag{1.11}$$

Proof. Let

$$Y_t := \mathbb{E}\left\{ \xi + \int_t^T f(s) ds \middle| \mathcal{F}_t \right\}, \tag{1.12}$$

for the martingale representation Theorem (A.3) there exists a stochastic process $Z \in L^2_{\mathcal{F}.}$ such that

$$Y(0) = \mathbb{E}\left\{\xi + \int_0^T f(s)ds \middle| \mathcal{F}_t\right\} - \int_0^t Z_s' dW_s, \tag{1.13}$$

so we have that (Y, Z) defined in (1.12) and (1.13) is the solution of (1.11), because $\xi \in L^2_T$

$$\mathbb{E}\left\{\xi + \int_0^T f(s)ds \middle| \mathcal{F}_T\right\} = \xi + \int_0^T f(s)ds = Y(0) + \int_0^T Z_s' dW_s, \tag{1.14}$$

matching (1.13) and(1.14) in $Y(0)$ it follows that

$$\int_t^T Z_s' dW_s = \xi + \int_0^T f(s)ds - \mathbb{E}\left\{\xi + \int_0^T f(s)ds \middle| \mathcal{F}_t\right\}. \tag{1.15}$$

Finally, substituting (1.12) and (1.15) in equation (1.11) verifies that (Y, Z) is the solution. □

Proposition 1.1. *Let $\xi \in L^2_T$ and $f : \Omega \times [0, T] \times \mathbb{R}^{n\times d} \to \mathbb{R}^d$ be the generator, that is $\mathcal{P} \otimes \mathcal{B}^{n\times d}$-measurable, that satisfies that $f(\cdot, 0) \in H^2_T$ and that is Lipschitz, i.e., exists a constant $c > 0$ such that*

$$|f(t, Z_1) - f(t, Z_2)| \leq c\,|Z_1 - Z_2|, \tag{1.16}$$

then there exists a unique pair $(Y, Z) \in H^2_T(\mathbb{R}^d) \times H^2_T(\mathbb{R}^{n\times d})$ which is the solution of

$$Y_t = \xi + \int_t^T f(s, Z_s)\,ds - \int_t^T Z_s' dW_s. \tag{1.17}$$

Proof. Uniqueness. Let (Y, Z) and $\left(\tilde{Y}, \tilde{Z}\right)$ be two solutions (1.17). Applying Itô's formula to $\left|Y_s - \tilde{Y}_s\right|^2$ for $s = t$ and $s = T$ it follows that

$$\begin{aligned}\left|Y_t - \tilde{Y}_t\right|^2 - \left|Y_0 - \tilde{Y}_0\right|^2 =& 2\int_0^t \left(Y_s - \tilde{Y}_s, f(s, Z_s) - f\left(s, \tilde{Z}_s\right)\right) ds + \\ &+ 2\int_0^t \left(Y_s - \tilde{Y}_s, Z_s - \tilde{Z}_s\right) dW_s + \int_0^t \left|Z_s - \tilde{Z}_s\right|^2 ds\end{aligned} \tag{1.18}$$

and

$$\underbrace{\left|Y_T - \tilde{Y}_T\right|^2}_{|\xi-\xi|^2=0} - \left|Y_0 - \tilde{Y}_0\right|^2 = 2\int_0^T \left(Y_s - \tilde{Y}_s, f(s, Z_s) - f\left(s, \tilde{Z}_s\right)\right) ds + \tag{1.19}$$

$$+ 2\int_0^T \left(Y_s - \tilde{Y}_s, Z_s - \tilde{Z}_s\right) dW_s + \int_0^T \left|Z_s - \tilde{Z}_s\right|^2 ds.$$

Matching (1.18) and (1.19) in $\left|Y_0 - \tilde{Y}_0\right|^2$ we get

$$\begin{aligned}\left|Y_t - \tilde{Y}_t\right|^2 + \int_t^T \left|Z_s - \tilde{Z}_s\right|^2 ds = &- 2\int_t^T \left(Y_s - \tilde{Y}_s, f(s, Z_s) - f\left(s, \tilde{Z}_s\right)\right) ds \\ &- 2\int_t^T \left(Y_s - \tilde{Y}_s, Z_s - \tilde{Z}_s\right) dW_s,\end{aligned} \tag{1.20}$$

also

$$\begin{aligned}-2\left(Y_s - \tilde{Y}_s, f(s, Z_s) - f\left(s, \tilde{Z}_s\right)\right) &= -2\left(\sqrt{2}c\left(Y_s - \tilde{Y}_s\right), \frac{f(s, Z_s) - f\left(s, \tilde{Z}_s\right)}{\sqrt{2}c}\right) \\ &= 2c^2\left|Y_s - \tilde{Y}_s\right|^2 + \frac{\left|f(s, Z_s) - f\left(s, \tilde{Z}_s\right)\right|^2}{2c^2} + \\ &\quad - \left|Y_s - \tilde{Y}_s + f(s, Z_s) - f\left(s, \tilde{Z}_s\right)\right| \\ &\leq 2c^2\left|Y_s - \tilde{Y}_s\right|^2 + \frac{\left|f(s, Z_s) - f\left(s, \tilde{Z}_s\right)\right|^2}{2c^2} \\ &\leq 2c^2\left|Y_s - \tilde{Y}_s\right|^2 + \frac{1}{2}\left|Z_s - \tilde{Z}_s\right|,^2 \quad \text{because } f \text{is Lipschitz.}\end{aligned} \tag{1.21}$$

From (1.20) and (1.21) the following inequality holds

$$\begin{aligned}\left|Y_t - \tilde{Y}_t\right|^2 + \int_t^T \left|Z_s - \tilde{Z}_s\right|^2 ds \leq &\int_t^T \left\{2c^2\left|Y_s - \tilde{Y}_s\right|^2 + \frac{1}{2}\left|Z_s - \tilde{Z}_s\right|^2\right\} ds \\ &- 2\int_t^T \left(Y_s - \tilde{Y}_s, Z_s - \tilde{Z}_s\right) dW_s.\end{aligned} \tag{1.22}$$

Taking expectation in (1.22)

$$\mathbb{E}\left|Y_t-\tilde{Y}_t\right|^2+\mathbb{E}\int_t^T\left|Z_s-\tilde{Z}_s\right|^2ds\leq 2c^2\mathbb{E}\int_t^T\left|Y_s-\tilde{Y}_s\right|^2ds+\frac{1}{2}\mathbb{E}\int_t^T\left|Z_s-\tilde{Z}_s\right|^2ds, \tag{1.23}$$

simplifying

$$\mathbb{E}\left|Y_t-\tilde{Y}_t\right|^2\leq-\frac{1}{2}\mathbb{E}\int_t^T\left|Z_s-\tilde{Z}_s\right|^2ds+2c^2\mathbb{E}\int_t^T\left|Y_s-\tilde{Y}_s\right|^2ds. \tag{1.24}$$

Applying Gronwall's Lemma (A.3) to (1.23)

$$0\leq\mathbb{E}\left|Y_t-\tilde{Y}_t\right|^2\leq-\frac{1}{2}\exp\left(2c^2\left(T-t\right)\right)\mathbb{E}\int_t^T\left|Z_s-\tilde{Z}_s\right|^2ds\leq 0, \tag{1.25}$$

i.e.,

$$0=\mathbb{E}\left|Y_t-\tilde{Y}_t\right|^2=-\frac{1}{2}\exp\left(2c^2\left(T-t\right)\right)\mathbb{E}\int_t^T\left|Z_s-\tilde{Z}_s\right|^2ds, \tag{1.26}$$

which prove the uniqueness of the solutions.

Existence. Let $(Y_0(t),Z_o(t))\equiv(0,0)$, $\{(Y_n(t),Z_n(t)):0\leq t\leq T\}_{n\geq1}$ be a sequence in $H_T^2\left(\mathbb{R}^d\right)\times H_T^2\left(\mathbb{R}^{n\times d}\right)$ defined recursively by

$$Y_n(t)=\xi+\int_t^T f(s,Z_{n-1}(s))ds-\int_t^T Z_n(s)dW_s, \tag{1.27}$$

which is well defined thanks to Lemma (1.1). Again, using Itô's formula (A.2) to $|Y_{n+1}(s)-Y_n(s)|^2$ for $s=t$ and $s=T$ and using the hypothesis over f that is Lipschitz as in (1.23) we obtain the inequality

$$\begin{aligned}\mathbb{E}\left|Y_{n+1}(t)-Y_n(t)\right|^2+\mathbb{E}\int_t^T\left|Z_{n+1}(s)-Z_n(s)\right|^2ds\leq&K\mathbb{E}\int_t^T\left|Y_{n+1}(t)-Y_n(t)\right|^2ds+\\&+\frac{1}{2}\mathbb{E}\int_t^T\left|Z_n(s)-Z_{n-1}(s)\right|^2ds,\end{aligned} \tag{1.28}$$

where $K=2c^2$.

Now, let $u_n(t) := \mathbb{E}\int_t^T |Y_n(t) - Y_{n-1}(t)|^2 \, ds$ and $v_n(t) := \mathbb{E}\int_t^T |Z_n(s) - Z_{n-1}(s)|^2 \, ds$, from (1.28) we deduce that

$$-\frac{d}{dt}\left(u_{n+1}(t)e^{Kt}\right) + e^{Kt}v_{n+1}(t) \leq \frac{1}{2}e^{Kt}v_n(t), \tag{1.29}$$

integrating (1.29) from t to T, and

$$-\underbrace{\int_t^T d\left(u_{n+1}(s)e^{Ks}\right)}_{-u_{n+1}(T)e^{KT}+u_{n+1}(t)e^{Kt}} + \int_t^T e^{Ks}v_{n+1}(s)ds \leq \int_t^T \frac{1}{2}e^{Ks}v_n(s)ds; \tag{1.30}$$

note that $u_n(T) = 0$, so that (1.30) is reduced to

$$u_{n+1}(t) + \int_t^T e^{K(s-t)}v_{n+1}(s)ds \leq \int_t^T \frac{1}{2}e^{K(s-t)}v_n(s)ds. \tag{1.31}$$

Let $\bar{c} := T\mathbb{E}\int_0^T |Z_1(t)|^2 \, dt = \sup_{0\leq t\leq T} v_1(t)$, then

$$\int_0^T e^{Kt}v_1(t)dt \leq \int_0^T e^{kT}\mathbb{E}\int_0^T |Z_1(s)|^2 \, dsdt = \bar{c}e^{kT}, \tag{1.32}$$

iterating the inequality (1.32), we obtain

$$\int_0^T e^{Kt}v_{n+1}(t)dt \leq \frac{1}{2^n}\bar{c}e^{kT}, \tag{1.33}$$

also from (1.31) we have

$$u_{n+1}(0) \leq \frac{1}{2^n}\bar{c}e^{kT}, \tag{1.34}$$

using the fact that $\frac{d}{dt}\left(u_{n+1}(t)\right) = -\mathbb{E}\left|Y_{n+1}(t) - Y_n(t)\right|^2 \leq 0$ is deduced from (1.29)

$$v_{n+1}(0) \leq \frac{1}{2}v_n(0) + Ku_{n+1}(0) \leq \frac{1}{2^n}\bar{K}e^{kT} + \frac{1}{2}v_n(0), \tag{1.35}$$

where $\bar{K} := \bar{c}Ke^{kT}$, it follows from (1.35)

$$v_{n+1}(0) \leq \frac{1}{2^n}\left(n\bar{K} + v_1(0)\right). \tag{1.36}$$

From (1.34) and (1.36) we have that $v_{n+1}(0)$ and $u_{n+1}(0)$ are summable, i.e., $(Y_n)_{n\geq 1} \in H_T^2\left(\mathbb{R}^d\right)$ y $(Z_n)_{n\geq 1} \in H_T^2\left(\mathbb{R}^{n\times d}\right)$ are of Cauchy and this is a complete space, so we can take limits when $n \to \infty$. Let

$$Y := \lim_{n\to\infty} Y_n \qquad Z := \lim_{n\to\infty} Z_n, \tag{1.37}$$

by construction (Y, Z) are solution of (1.17). □

Theorem 1.1. *Let $\xi \in L_T^2$ and $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \to \mathbb{R}^d$ be the generator that is $\mathcal{P} \otimes \mathcal{B}^d \otimes \mathcal{B}^{n\times d}$-measurable that satisfies $f(\cdot,0,0) \in H_T^2$ and is Lipschitz, i.e., exists a constant $c > 0$ such that*

$$|f(t,Y_1,Z_1) - f(t,Y_2,Z_2)| \leq c\left(|Y_1 - Y_2| + |Z_1 - Z_2|\right), \tag{1.38}$$

then there exists a unique pair $(Y,Z) \in H_T^2\left(\mathbb{R}^d\right) \times H_T^2\left(\mathbb{R}^{n\times d}\right)$ that is the solution of

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,ds - \int_t^T Z_s' dW_s. \tag{1.39}$$

Proof. Let (Y,Z) and $\left(\tilde{Y},\tilde{Z}\right)$ be two solutions of (1.17). By a similar argument to Proposition 1.1 the following is obtained

$$\mathbb{E}\left|Y_t - \tilde{Y}_t\right|^2 + \mathbb{E}\int_t^T \left|Z_s - \tilde{Z}_s\right|^2 ds = -2\int_t^T \left(Y_s - \tilde{Y}_s, f(s,Y_s,Z_s) - f\left(s,\tilde{Y}_s,\tilde{Z}_s\right)\right) ds, \tag{1.40}$$

since f is Lipschitz

$$\mathbb{E}\left|Y_t - \tilde{Y}_t\right|^2 + \mathbb{E}\int_t^T \left|Z_s - \tilde{Z}_s\right|^2 ds \leq 2c^2\mathbb{E}\int_t^T \left|Y_s - \tilde{Y}_s\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left|Z_s - \tilde{Z}_s\right|^2 ds; \tag{1.41}$$

and simplifying

$$\mathbb{E}\left|Y_t - \tilde{Y}_t\right|^2 \leq -\frac{1}{2}\mathbb{E}\int_t^T \left|Z_s - \tilde{Z}_s\right|^2 ds + 2c^2\mathbb{E}\int_t^T \left|Y_s - \tilde{Y}_s\right|^2 ds. \tag{1.42}$$

Applying Gronwall's Lemma A.3 to (1.40), we deduce

$$0 \leq \mathbb{E}\left|Y_t - \tilde{Y}_t\right|^2 \leq -\frac{1}{2}\mathbb{E}\int_t^T \left|Z_s - \tilde{Z}_s\right|^2 ds \exp\left(2c^2(T-t)\right) \leq 0, \tag{1.43}$$

from which it follows the uniqueness of the solutions.

Existence. Let $(Y_0(t), Z_o(t)) \equiv (0,0)$, $\{(Y_n(t), Z_n(t)) : 0 \leq t \leq T\}_{n\geq 1}$ be a sequence in $H_T^2\left(\mathbb{R}^d\right) \times H_T^2\left(\mathbb{R}^{n\times d}\right)$ defined recursively by

$$Y_n(t) = \xi + \int_t^T f(s, Y_{n-1}(s), Z_n(s))ds - \int_t^T Z_n(s)dW_s, \tag{1.44}$$

which is well defined thanks to Proposition (1.1). Using the same idea as in the last proof is obtained

$$\mathbb{E}\left|Y_{n+1}(t)-Y_n(t)\right|^2+\frac{1}{2}\mathbb{E}\int_t^T\left|Z_{n+1}(s)-Z_n(s)\right|^2 ds \leq c\mathbb{E}\int_t^T\left|Y_{n+1}(t)-Y_n(t)\right|^2 ds + \\ + c\mathbb{E}\int_t^T\left|Y_n(s)-Y_{n-1}(s)\right|^2 ds. \tag{1.45}$$

Defined $u_n(t) := \mathbb{E}\int_t^T \left|Y_n(t)-Y_{n-1}(t)\right|^2 ds$ of (1.45) we deduce

$$-\frac{d}{dt}u_{n+1}(t)-cu_{n+1}(t)\leq cu_n(t). \quad u_{n+1}(T)=0, \tag{1.46}$$

integrating (1.46), we have

$$u_{n+1}(t)\leq c\int_t^T \frac{1}{2}e^{c(s-t)}u_n(s)ds. \tag{1.47}$$

Finally, iterating (1.47) leads

$$u_{n+1}(0)\leq \frac{\left(ce^{cT}\right)^n}{n!}u_1(0), \tag{1.48}$$

In (1.48) we have that $u_{n+1}(0)$ converges, and joined to (1.45) we conclude that $(Y_n)_{n\geq 1}\in H_T^2\left(\mathbb{R}^d\right)$ and $(Z_n)_{n\geq 1}\in H_T^2\left(\mathbb{R}^{n\times d}\right)$ are Cauchy sequences, so we can take limits when $n\to\infty$. Define

$$Y := \lim_{n\to\infty} Y_n \qquad Z := \lim_{n\to\infty} Z_n, \tag{1.49}$$

by construction (Y,Z) is the solution of (1.38). □

1.5 One dimensional linear BSDEs

This kind of equations is very useful because we can give explicitly the component Y of the solution.

Proposition 1.2. *Solution of a linear BSDE. Let (β,μ) be a bounded $\left(\mathbb{R},\mathbb{R}^d\right)$-valued progressively measurable process, $\varphi\in H_T^2(\mathbb{R})$ and $\xi\in L_T^2(0,T)$. We consider the following linear BSDE:*

$$-dY_t=(\varphi_t+Y_t\beta_t+Z_t\mu_t)\,dt-Z_t dW_t; \quad Y_T=\xi. \tag{1.50}$$

1. The equation has a unique solution (Y, Z) and Y is given explicitly by

$$Y_t = \mathbb{E}\left[\xi\Gamma_{t,T} + \int_t^T \Gamma_{t,s}\varphi_s ds \middle| \mathcal{F}_t\right], \tag{1.51}$$

where $(\Gamma_{t,s})_{s\geq t}$ is the adjoint process defined by the forward linear SDE

$$d\Gamma_{t,s} = \Gamma_{t,s}\left(\beta_s ds + \mu_s dW_s\right); \quad \Gamma_{t,t} = 1, \tag{1.52}$$

satisfying the flow property $\forall t \leq s \leq u$, $\Gamma_{t,s}\Gamma_{s,u} = \Gamma_{t,u}$ $\mathbb{P}$-a.s.

2. If ξ and φ are both non-negative, then the process $(Y_t)_{t\leq T}$ is non-negative. Moreover, if in addition $Y_t = 0$ on $B \in \mathcal{F}_t$, then $\mathbb{P}$-a.s. on B for any $s \geq t$, $Y_s = 0$, $\xi = 0$ and $\varphi_s = 0$, $Z_s = 0$ $d\mathbb{P} \otimes ds$-a.e.

1.6 Comparison Theorem

Stochastic domination theorems plays an important role in the theory of stochastic processes as well as their applications. The results of the theory of SDE, establishing pathwise almost surely dominance, i.e., when one process with probability one is greater than or equal to another, are referred to as the comparison theorems.

These types of theorems are used in a wide range of mathematical problems: from existence and uniqueness of solution of SDE's to asymptotic behavior. A comparison theorem for solutions of stochastic differential equations was discussed first by A.V. Skorohod [14] and T. Yamada [16]. Later it was modified by N. Ikeda and S. Watanabe [6] for its application to some stochastic optimal control problem.

Theorem 1.2. *Comparison Theorem. Let (Y, Z) and $\left(\tilde{Y}, \tilde{Z}\right)$ be the solutions of two BSDEs with associated parameters (g, ξ) and $\left(\tilde{g}, \tilde{\xi}\right)$. Suppose that $\{g(t, 0, 0)\}_{t\leq T} \in H_T^2$, g is Lipschitz continuous with constant C and $\widetilde{g}\left(t, \tilde{Y}_t, \tilde{Z}_t\right)$ has to be an element of H_T^2. If $\xi \leq \tilde{\xi}$ $\mathbb{P}$-a.s. and $g\left(t, \tilde{Y}_t, \tilde{Z}_t\right) \leq \tilde{g}\left(t, \tilde{Y}_t, \tilde{Z}_t\right)$ $dt \otimes d\mathbb{P}$-a.e., then*

$$Y_t \leq \tilde{Y}_t, \quad \forall t \in [0, T] \ \ \mathbb{P} - a.s. \tag{1.53}$$

Remark 1.1. Moreover, the comparison is strict, i.e., if in addition $Y_0 = \tilde{Y}_0$, then $\xi = \tilde{\xi}$, $g\left(t, \tilde{Y}_t, \tilde{Z}_t\right) \leq \tilde{g}\left(t, \tilde{Y}_t, \tilde{Z}_t\right)$ and $Y_t = \tilde{Y}_t$, $\forall t \in [0, T]$ $\mathbb{P}$−a.s. In particular, whenever either $\mathbb{P}\left(\xi < \tilde{\xi}\right) > 0$ or $g\left(t, \tilde{Y}_t, \tilde{Z}_t\right) < \tilde{g}\left(t, \tilde{Y}_t, \tilde{Z}_t\right)$ on a set of positive $dt \otimes d\mathbb{P}$-measure, then $Y_0 < \tilde{Y}_0$.

Proof. By hypothesis $-\tilde{g}\left(t, \tilde{Y}_t, \tilde{Z}_t\right) \leq -g\left(t, \tilde{Y}_t, \tilde{Z}_t\right)$, then using the Lipschitz property of g we have that

$$g\left(t, Y_t, Z_t\right) - g\left(t, \tilde{Y}_t, \tilde{Z}_t\right) \leq C\left(\left|Y_t - \tilde{Y}_t\right| + \left|Z_t - \tilde{Z}_t\right|\right). \tag{1.54}$$

Applying Itô's formula A.2 and Tanaka's formula A.2 to $e^{\alpha t}\left[\left(Y_t - \tilde{Y}_t\right)^+\right]^2$, for any $\alpha > 0$ we obtain

$$\begin{aligned} e^{\alpha t}\left[\left(Y_t - \tilde{Y}_t\right)^+\right]^2 =& e^{\alpha T}\left[\left(\xi - \tilde{\xi}\right)^+\right]^2 - \int_t^T \alpha e^{\alpha s}\left[\left(Y_s - \tilde{Y}_s\right)^+\right]^2 ds + \\ & - 2\int_t^T e^{\alpha s}\left(Y_s - \tilde{Y}_s\right)^+ d\left(Y_s - \tilde{Y}_s\right)^+ + \\ & - \frac{1}{2}\int_t^T e^{\alpha s}\left[1_{\left(Y_s - \tilde{Y}_s\right)^+}\right] d\left\langle Y_{\cdot} - \tilde{Y}_{\cdot}\right\rangle_s . \end{aligned} \tag{1.55}$$

Simplifying and using the hypothesis $\xi \leq \tilde{\xi}$ $\mathbb{P}$-a.s.,

$$e^{\alpha t}\left[\left(Y_t - \tilde{Y}_t\right)^+\right]^2 = -2\int_t^T e^{\alpha s}\left(Y_s - \tilde{Y}_s\right)^+\left(Z_s - \tilde{Z}_s\right) dW_s + \int_t^T V_s ds, \quad \mathbb{P} - \text{a.s.} \tag{1.56}$$

where:

$$\begin{aligned} V_s = & e^{\alpha s}\left\{-\alpha\left[\left(Y_s - \tilde{Y}_s\right)^+\right]^2 - 1_{\left\{Y_s > \tilde{Y}_s\right\}}\left|Z_s - \tilde{Z}_s\right|^2\right\} + \\ & + e^{\alpha s}\left\{2\left(Y_s - \tilde{Y}_s\right)^+\left[g\left(s, Y_s, Z_s\right) - \tilde{g}\left(s, \tilde{Y}_s, \tilde{Z}_s\right)\right]\right\}, \quad \mathbb{P} - a.s. \end{aligned} \tag{1.57}$$

and then from (1.54)

$$\begin{aligned} V_s \leq & e^{\alpha s}\left\{-\alpha\left[\left(Y_s - \tilde{Y}_s\right)^+\right]^2 - 1_{\left\{Y_s > \tilde{Y}_s\right\}}\left|Z_s - \tilde{Z}_s\right|^2\right\} + \\ & e^{\alpha s}\left\{2C\left(Y_s - \tilde{Y}_s\right)^+\left[\left|Y_t - \tilde{Y}_t\right| + \left|Z_t - \tilde{Z}_t\right|\right]\right\} \quad \mathbb{P} - a.s. \end{aligned} \tag{1.58}$$

By the polarization formula $-\alpha a^2+2Cab=-\alpha\left(a-\frac{C}{a}b\right)^2+\frac{C^2}{\alpha}b^2\leq\frac{C^2}{\alpha}b^2$, for any $a,b\in\mathbb{R}$ we obtain:

$$-\alpha\left(a^+\right)^2-1_{\{a>0\}}b^2+2Ca^+\left(|a|+|b|\right)=-1_{\{a>0\}}\left[-\alpha\,|a|^2-|b|^2+2C\,|a|\left(|a|+|b|\right)\right] \tag{1.59}$$

$$=1_{\{a>0\}}\left[\left(C^2+2C-\alpha\right)|a|^2-\left(|b|-C\,|a|\right)^2\right] \tag{1.60}$$

$$\leq 0\ \ \forall\alpha\geq C^2+2C. \tag{1.61}$$

Taking $a=Y_s-\tilde{Y}_s$ and $b=Z_s-\tilde{Z}_s$ in (1.58),

$$V_s\leq 0, \tag{1.62}$$

and then, for $\alpha\geq C^2+2C$, from (1.56)

$$e^{\alpha t}\left[\left(Y_t-\tilde{Y}_t\right)^+\right]^2\leq -2\int_t^T e^{\alpha s}\left(Y_s-\tilde{Y}_s\right)^+\left(Z_s-\tilde{Z}_s\right)dW_s\ \ \forall t\in[0,T]\ \ \mathbb{P}-a.s. \tag{1.63}$$

Applying the expected value

$$\mathbb{E}\left[e^{\alpha t}\left[\left(Y_t-\tilde{Y}_t\right)^+\right]^2\right]\leq 0\ \ \forall t\in[0,T]\ \ \mathbb{P}-a.s. \tag{1.64}$$

Hence

$$Y_t\leq\tilde{Y}_t,\quad\forall t\in[0,T]\ \ \mathbb{P}-a.s. \tag{1.65}$$

□

1.7 Applications

This section shows some of the various applications of BSDE, which consider the filtered probability space $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$. The first application shown is a stochastic optimal control problem, which its solution can be found solving a system of stochastic differential equations. No tackle in the details for this example, because in Chapter 3 we detail the stochastic optimal control problem. The second example is an application to finance in the area of option pricing and contingent assets for European options. The third example is a problem that comes from economy, and is a generalization of the concept of recursive utility.

1.7.1 A Stochastic Optimal Control Problem

The stochastic optimal control is used to solve optimization problems in random systems that evolve over time and are likely to be influenced by external forces. It consists mainly of a dynamic forward and a functional optimized. In this example, to be consider the following dynamic stochastic differential equation given by the controlled process

$$\begin{cases} dX(t) = [aX(t) + bu(t)]\,dt + dW(t), & t \in [0,T] \\ X(0) = x, \end{cases} \tag{1.66}$$

where W is a brownian motion, $X = \{X(t) :\ 0 \leq t \leq T\}$ is called the state process taking values in $(S, \mathcal{B}(S))$, where S is a Polish space, that is, a closed and bounded set of $\mathbb{R}^n$, and $u = \{u(t),\ 0 \leq t \leq T\}$ is the control process that takes values in $\mathcal{U}$, which is the admissible control set. The state at time t is represented by $X(t)$ while the control at time t is given by $u(t)$. We consider processes (X, u) to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and square integrable.

For simplicity, for this example, consider X, u and W one-dimensional and a, b constants.

We will consider a cost functional given by

$$J(u) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[|X(t)|^2 + |u(t)|^2\right] dt + |X(T)|^2\right\}. \tag{1.67}$$

In this case, the optimal control problem is to minimize the value of the functional (1.67) subject to the equation of state (1.66); one can show that there is a control $u \in \mathcal{U}$ that minimizes this functional and it is unique a.s.

Suppose that u is the optimal control and X is the corresponding process states. Then, for any admissible controls $v \in \mathcal{U}$ (ie a process $\{\mathcal{F}_t\}_{t\geq 0}$-adapted square integrable), we have

$$0 \leq \frac{J(u+\epsilon v) - J(u)}{\epsilon} \tag{1.68}$$

$$= \frac{\frac{1}{2}\mathbb{E}\left\{\int_0^T \left[\left|\widetilde{X}_t\right|^2 + |u_t + \epsilon v_t|^2\right] dt + \left|\widetilde{X}_T\right|^2\right\} - \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[|X_t|^2 + |u_t|^2\right] dt + |X_T|^2\right\}}{\epsilon}, \tag{1.69}$$

where $\widetilde{X}$ has the following dynamic

$$\begin{cases} d\widetilde{X}_t = \left\{a\widetilde{X}_t + b\left[u_t + \epsilon v_t\right]\right\} dt + dW_t, \quad t \in [0,T] \\ \widetilde{X}_0 = x. \end{cases} \tag{1.70}$$

It follows, from (1.69),

$$0 \leq \frac{\frac{1}{2}\mathbb{E}\left\{\int_0^T \left[\left|\widetilde{X}_t\right|^2 - |X_t|^2 + 2\epsilon u_t v_t + \epsilon^2 v_t^2\right] dt + \left|\widetilde{X}_T\right|^2 - |X_T|^2\right\}}{\epsilon} \tag{1.71}$$

$$= \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[\left(\widetilde{X}_t + X_t\right)\frac{\left(\widetilde{X}_t - X_t\right)}{\epsilon} + 2u_t v_t + \epsilon v_t^2\right] dt + \left(\widetilde{X}_T + X_T\right)\frac{\left(\widetilde{X}_T - X_T\right)}{\epsilon}\right\}. \tag{1.72}$$

Defining $\xi = \lim_{\epsilon\to 0}\frac{(\widetilde{X}_t - X_t)}{\epsilon}$, from (1.66) and (1.70) it follows that ξ satisfies the following variational system:

$$\begin{cases} d\xi(t) = [a\xi(t) + bv(t)]\, dt, \quad t \in [0,T] \\ \xi(0) = 0. \end{cases} \tag{1.73}$$

Letting $\epsilon \to 0$ in (1.72),

$$0 \leq \mathbb{E}\left\{\int_0^T [X_t\xi_t + u_t v_t]\, dt + X_T\xi_T\right\}. \tag{1.74}$$

For more information about (1.68), we introduce the BSDE

$$\begin{cases} dY(t) = -\left[aY(t) + X(t)\right] dt + Z(t)dW(t), \quad t \in [0,T] \\ Y(T) = X(T). \end{cases} \tag{1.75}$$

Suppose that (1.75) admits a unique adapted solution (Y, Z). Then, applying Itô's formula to $Y(t)\xi(t)$ we obtain

$$\begin{aligned}\mathbb{E}\left[X(T)\xi(T)\right] =&\mathbb{E}\left[Y(T)\xi(T)\right] \\ =&\mathbb{E}\int_0^T \left\{\left[-aY(t) - X(t)\right]\xi(t) + Y(t)\left[a\xi(t) + bv(t)\right]\right\} dt \\ =&\mathbb{E}\int_0^T \left\{-X(t)\xi(t) + Y(t)bv(t)\right\} dt.\end{aligned} \tag{1.76}$$

Because of (1.74) we reach

$$0 \leq \mathbb{E}\int_0^T \left[bY(t) + u(t)\right] v(t)dt. \tag{1.77}$$

As v is arbitrary, we arrive to

$$u(t) = -bY(t), \quad c.d.s.\ t \in [0, T]. \tag{1.78}$$

Note that since Y is part of the solution of the BSDE of (1.75) and is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, so that u is an admissible control. Substituting (1.78) in (1.66) we arrive at the following optimality system:

$$\begin{cases} dX(t) = \left[aX(t) + b^2Y(t)\right]dt + dW(t) & t \in [0, T] \\ dY(t) = -\left[aY(t) + X(t)\right]dt + Z(t)dW(t), & t \in [0, T] \\ X(0) = x, \quad Y(T) = X(T). \end{cases} \tag{1.79}$$

Finally, we have a FSDE of X (as it involves an initial condition), while for Y we have a BSDE, which is why (1.79) is known as a Backward Forward Stochastic Differential Equation (BFSDE). If it is shown that (1.79) admits a unique solution adapted (X, Y, Z), then (1.78) gives an optimal control that solves the original problem.

1.7.2 Option Pricing and Contingent Claims Valuation

Consider a complete market with a riskless bond and an asset. Suppose that prices are subject to the following system of SDE:

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt & \text{Bond} \\ dP(t) = P(t)b(t)dt + P(t)\sigma(t)dW(t) & \text{Stock,} \end{cases} \tag{1.80}$$

where r is the bond's interest rate, b is the rate of appreciation and σ the volatility of the stock.

A financial option is a derivative financial instrument that is set in a contract that gives its buyer the right but not the obligation, to buy or sell assets or securities (the underlying asset can be stocks, bonds, etc.,) at a predetermined price q (strike or exercise price) to a specific date T (maturity).

This example is particularly interested in European options, in which the time of exercise of the option is specified and is equal to T . We will take the European call and will assume that the decision of the holder to exercise its option or not will depend only on $P(T)$, i.e. the stock price at time T. For this case, the gain of the holder will be $(P(T)-q)^+$, which is a random variable $\mathcal{F}_T-$measurable.

The problem of the valuation of options is how to determine a premium for this contract at time $t=0$. In general, it is called an option contract if payment at time $t=T$ can be written explicitly as a function of $P(T)$, e.g. $(P(T)-q)^+$. In all other cases where payment of the contract at time $t=T$ is just a random variable $\mathcal{F}_T-$measurable and is called a contingent claim.

Let $Y(t)$ be the agent's wealth at time t. Suppose that the agent sells the option at price y at $t=0$, i.e. $Y(0)=y$. Then, the agent invests in the market a portion of his wealth $\pi(t)$, called a portfolio, into the stock, and the rest $(Y(t)-\pi(t))$ in the bond. We also assume that the agent can spend his wealth, denote all that has spent up to time t by the function $C(t)$. This process is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and nondecreasing; one can show that the dynamics of the wealth of the owner Y and portfolio and consumption processes (π, C) can be expressed by the SDE:

$$\begin{cases} dY(t)=\{r(t)Y(t)+Z(t)\theta(t)\}\,dt+Z(t)dW(t)-dC(t) \\ Y(0)=y, \end{cases} \tag{1.81}$$

where $Z(t)=\pi(t)\sigma(t)$, $\theta:=\sigma^{-1}(t)\,[b(t)-r(t)]$, called risk premium process. The objective for the agent is to choose a pair (π, C) such that for any contingent asset $H\in L^2_{\mathcal{F}_T}$ compliance $Y(T)\geq H$. In the event that there is a pair (π, C) is called a hedging strategy against H. The fair price of this contingent claim is the smallest initial endowment for winch the hedging strategy exists, i.e.

$$y^*=\inf\left\{y=Y(0);\ \exists\,(\pi, C)\,,\ \text{tal que } Y^{\pi,C}(T)\geq H\right\}. \tag{1.82}$$

Now suppose that the agent is very cautious and does not spend anything, that is, $C \equiv 0$ and can choose a portfolio π such that $Y(T) = H$, and H can be written explicitly in terms of price $P(T)$, $(H = g(P(T)))$. Under these assumptions we have that (1.80) and (1.81) can be seen as:

$$\begin{cases} dP(t) = P(t)b(t)dt + P(t)\sigma(t)dW(t) \\ dY(t) = \{r(t)Y(t) + Z(t)\theta(t)\}\, dt + Z(t)dW(t) \\ P(0) = p, \quad Y(T) = g(P(T)), \end{cases} \tag{1.83}$$

which is a BSDE. An interesting result is that if (1.83) admits a solution (Y, Z). Then the pair$(\pi, 0)$, where $\pi = Z\sigma^{-1}$, is an optimal strategy of hedging and $y = Y(0)$ is the right price.

1.7.3 Stochastic differential utility

Recursive methods have become a standard tool for the study of economic behavior in stochastic dynamic environments. In this example, we characterize the class of preferences that is the natural complement to this framework, namely recursive utility. The main idea of this model comes from the following question, why model preferences rather than behavior? Preferences plays two critical roles in economic models. First, preferences provide, in principle, an unchanging feature of a model in which agents can be confronted with a wide range of different environments, institutions, or policies. For each environment, we derive behavior (decision rules) from the same preferences.

If we modeled behavior directly, we would also have to model how it is adjusted to changing circumstances. The second role played by preferences is to allow us to evaluate the welfare effects of changing policies or circumstances. Without the ranking of opportunities that a model of preferences provides, it's not clear how we should distinguish good policies from bad.

As we will see, this logic applies equally well to environments in which current actions affect the values of random events for all future periods. In this case, the two-period tradeoff is between current utility and a certainty equivalent of random future utility. This recursive approach not only allows complicated dynamic optimization problems to be characterized as much simpler and more intuitive two-period problems, it also lends itself to straightforward computational methods. Since many computational algorithms for solving stochastic dynamic models themselves rely on recursive methods, numerical versions of recursive utility models can be solved and simulated using standard algorithms.

This application is an extension of the notion of recursive utility in continuous time with a stochastic approach. In the discrete case the problem is to find some utility functions that satisfy the recursion relation. For example, assume that the consumption plan is denoted by $c = \{c_0, c_1, \dots\}$, where c_t represents consumption at time t, while the utility at time t is given by V_t. It says that $V = \{V_t : t = 0, 1, \dots\}$ defines a recursive utility if the sequence $V_0, V_1, \dots$ satisfies the recursive relation:

$$V_t = W\left(c_t, V_{t+1}\right), \tag{1.84}$$

where the function W is called the aggregator. Note that (1.84) is a backward recursion.

For the case in continuous time the scheme is denoted by $c = \{c(t) : \ t \geq 0\}$, where $c(t) \geq 0 \ \forall t \geq 0$, while the value at time t is given by $Y(t) := U\left(\{c(s) : \ s \geq t\}\right)$ and the recursion (1.84) is replaced by the differential equation:

$$\frac{dY(t)}{dt} = -f\left(c(t), Y(t)\right), \tag{1.85}$$

with f the aggregator. If the solution (1.85) can be determined, then $U(c) = Y(0)$ defines the utility function.

A variation of (1.84) and of (1.85) is the finite horizon case, i.e. that there is a terminal time $T > 0$, such that the problem is restricted to $0 \leq t \leq T$. Suppose that the utility of terminal consumption is given by $u\left(c\left(T\right)\right)$ for some utility function u given. Then the differential equation with terminal condition $Y(T) = u\left(c\left(T\right)\right)$ is

$$Y(t) = u\left(c\left(T\right)\right) + \int_t^T f\left(c(s), Y(s)\right) ds, \quad t \in [0, T]. \tag{1.86}$$

The stochastic model assumes that Y and c are stochastic processes defined on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ and that these processes are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted. Taking the conditional expectation in both sides of (1.86), we have

$$Y(t) = \mathbb{E}\left[Y(t)\,|\mathcal{F}_t\right] = \mathbb{E}\left\{u\left(c\left(T\right)\right) + \int_t^T f\left(c(s), Y(s)\right) ds\,|\mathcal{F}_t\right\}, \tag{1.87}$$

for all $t \in [0,T]$. In the special case where the filtration is generated by a brownian motion, we can use the martingale representation Theorem A.3 to justify that there is a stochastic process Z such that

$$Y(t) = u\left(c\left(T\right)\right) + \int_t^T f\left(c(s), Y(s)\right) ds - \int_t^T Z(s) dW(s), \quad t \in \left[0, T\right], \tag{1.88}$$

is a BSDE.

Chapter 2

Quadratic Case

Section 1.4 studies the theorems of existence and uniqueness of the solution of a BSDE for the case where the generator is Lipschitz. But what happens when the generator does not satisfy the Lipschitz condition?. Is there a solution and if it exists, is it unique? In this chapter, we will consider the case in which the generator increases quadratically in the variable z, i.e.

$$\left|f\left(\omega,t,y_1,z_1\right)-f\left(\omega,t,y_2,z_2\right)\right|\leq C\left(\left|y_1-y_2\right|+\left|z_1-z_2\right|^2\right)\ \ \forall(y_1,z_1),\ (y_2,z_2)\quad d\mathbb{P}\otimes dt\ \text{ c.s.} \tag{2.1}$$

This chapter shows classic tools to prove the existence and uniqueness of the solution of partial differential equations, which is also useful for the SDE. With this tool we give sufficient conditions for the existence of at least one solution of a BSDE in the case where the generator grows quadraticly. Also we give a theorem of the stability of the solutions that give us an idea of dimensions of the solution and then we can make numerical approximations to solutions.

2.1 Existence and uniqueness

Consider the following BSDE

$$Y_t=\xi+\int_t^T\frac{1}{2}\left|Z_s\right|^2ds-\int_t^TZ_sdW_s,\quad\forall t\in\left[0,T\right]. \tag{2.2}$$

If $y_t=e^{Y_t}$, from the Itô's formula

$$y_t=e^{\xi}-\int_t^Ty_tdY_s-\int_t^Ty_td\left\langle Y\right\rangle_s,\quad\forall t\in\left[0,T\right], \tag{2.3}$$

writting

$$y_t = e^{\xi} - \int_t^T y_s \left(-\frac{1}{2} |Z_s|^2 ds + Z_s dW_s \right) - \frac{1}{2} \int_t^T y_s |Z_s|^2 ds, \quad \forall t \in [0,T], \tag{2.4}$$

we get

$$y_t = e^{\xi} - \int_t^T z_s dW_s, \quad \forall t \in [0,T], \tag{2.5}$$

where $z_s = y_s Z_s$. As (2.5) is an equation that is solvable with a unique solution because of Lemma 1.1, when $e^{\xi} \in L_T^2$, this is a way to ensure that (2.2) has a unique solution when $\xi \in L_T^{\infty}$. This solution is given explicitly by

$$y_t = \mathbb{E}\left(e^{\xi} \middle| \mathcal{F}_t \right) \quad \forall t \in [0,T], \tag{2.6}$$

while the process z is given by the martingale representation Theorem A.3.

Thanks to the bijection of the exponential function, equation (2.3) must have a solution that is unique, and it is given by

$$Y_t = \ln y_t, \quad Z_t = \frac{z_t}{y_t} \quad \forall t \in [0,T]; \tag{2.7}$$

note that Z_t is well defined as

$$\begin{aligned} y_t &= \mathbb{E}\left(e^{\xi} \middle| \mathcal{F}_t \right) & (2.8)\\ &\geq e^{-|\xi|_{\infty}} & (2.9)\\ &> 0 \text{ a.s.} & (2.10) \end{aligned}$$

2.2 A priori estimates and existence

Another way to prove the existence of the solution is by a priori estimation. In the theory of Partial Differential Equation (PDE) and SDE, an a priori estimate is an estimate for the size of a solution or its derivatives of a PDE. One reason for their importance is that if one can prove an a priori estimate for solutions of a differential equation, then it is often possible to prove that solutions exist.

The proofs of the following theorems can be found in the article of Kobylanski (2000)[8].

Theorem 2.1. *Existence. Let $\alpha_0, \beta_0, b \in \mathbb{R}$, $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function, (f, ξ) the data of a BSDE, $\xi \in L_T^\infty$ and f that satisfies*

$$f(t, y, z) = A(t, y, z)\, y + f_0(t, y, z), \tag{2.11}$$

with

$$\beta_0 \leq A(t, y, z) \leq \alpha_0 \quad a.s. \tag{2.12}$$

$$|f_0(t, y, z)| \leq b + c(|y|)\, |z|^2 \quad a.s., \tag{2.13}$$

for all $(t, y, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$. Then the BSDE has at least one solution (Y, Z). Moreover there exists a minimal solution (Y_*, Z_*)(resp. a maximal solution (Y^*, Z^*)) such that for any set of data (g, ζ) of BSDE, if

$$f \leq g \quad and \quad \xi \leq \zeta \quad (resp.\ f \geq g \quad and \quad \xi \geq \zeta) \tag{2.14}$$

and for any solution (Y_g, Z_g) of the BSDE with data (g, ζ),

$$Y_* \leq Y_g \quad (resp.\ Y^* \geq Y_g). \tag{2.15}$$

The idea of the proof consists first of an exponential change $Y = e^{2Cy}$ in order to control its growth in z. Then it uses a truncation argument in order to control the growth of f in y by defining a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \to f$ and f_n satisfies the Lipschitz condition over y. Therefore, applying *Proposition 2.1*, the process $\{Y_n\}$ converges uniformly to Y and there exists Z, such that a subsequence of $\{Z_n\}$ converges to Z, where (Y_n, Z_n) is the solution of the BSDE with data (f_n, ξ).

Proposition 2.1. *Monotone stability. Let (f, ξ) the data of a BSDE and $\{(f_n, , \xi_n), n \in \mathbb{N}\}$ be a sequence of data such that:*

1. *The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f locally uniformly on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$, for each $n \in \mathbb{N}$, $\xi_n \in L_T^\infty$ and $\{\xi_n\}_{n\in\mathbb{N}}$ converges to ξ in L_T^∞.*
2. *There exists $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $T > 0$, $k \in L_T^1$, and there exists $C > 0$ such that*

$$\forall n \in \mathbb{N},\ \forall (t, y, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d, \quad |f_n(t, y, z)| \leq k_t + C\, |z|^2. \tag{2.16}$$

3. *For each n, the BSDE with data (f_n, ξ_n) has a solution (Y_n, Z_n) such that the sequence $\{Y_n\}_{n\in\mathbb{N}}$ is monotonic, and there exists $M > 0$ such that for all $n \in \mathbb{N}$, $\|Y_n\|_\infty \leq M$.*

Then there exists a pair of processes (Y, Z) *such that for all* $T \in \mathbb{R}^+$,

$$\lim_{n\to\infty} Y_n = Y \quad \textit{uniformly on } [0,T] \tag{2.17}$$

$$\{Z_n\}_{n\in\mathbb{N}} \textit{ converges to } Z \textit{ in } H_T^2 \tag{2.18}$$

and (Y, Z) *is the solution of the BSDE with data* (f, ξ).

Remark. The limit coefficient f satisfies the assumptions of quadratic growth, but not necessarily a comparison principle. Hence the solution we find here might not be unique.

The idea of the demonstration is first to show that exists a constant K such that, for all $n \in \mathbb{N}$

$$\mathbb{E}\left(\int_0^T |Z_s|^2 \, ds\right) \le K. \tag{2.19}$$

Therefore, there exists a process $Z \in H_T^2(\mathbb{R}^d)$ and a subsequence $\{Z^{n_j}\}_j$ of $\{Z_n\}_n$ such that $\{Z^{n_j}\}_j \to \{Z_n\}_n$ weakly in $H_T^2(\mathbb{R}^d)$. The point is to show that in fact the whole sequence converges strongly to Z in $H_T^2(\mathbb{R}^d)$.

2.3 Comparison theorem and uniqueness

The comparison theorems help us to prove uniqueness on BSDE. The first comparison Theorem 1.2 was for the case the generator satisfies the Lipschitz condition. We will give the comparison theorem in case which the generator has a quadratic growth. This theorem was prove by the first time by Kobylanski [8].

Definition 2.1. Supersolution and subsolution of a BSDE. A supersolution (resp. a subsolution) of a BSDE with coefficient f and terminal condition ξ is an adapted process $(Y_t, Z_t, C_t)_{t\in[0,T]}$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s + \int_t^T dC_s, \tag{2.20}$$

$$(\text{resp.}) Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s - \int_t^T dC_s, \tag{2.21}$$

where $(C_t)_{t\in[0,T]}$ is a right continuous increasing process $C \in RCI$ and where, in the classical framework (i.e ., when f is Lipschitz continuous in Y and Z, and when ξ is square integrable), the process $(Y_t, Z_t)_{t\in[0,T]}$ is assumed to be square integrable. Because of the quadratic growth , we will asume here that $\{Y_t\}_{t\in[0,T]}$ is a one dimensional bounded process and $\{Z_t\}_{t\in[0,T]}$ is a square integrable process.

Assumption 2. *The generator f satisfies for all $t \geq 0,\ u \in [-M, M]$ and $z \in \mathbb{R}^d$*

$$|f(t,u,z)| \leq I(t) + C|z|^2 \quad a.s. \tag{2.22}$$

$$\left|\frac{\partial f}{\partial z}(t,u,z)\right| \leq k(t) + C|z| \quad a.s. \tag{2.23}$$

for $I \in L^1,\ k \in L^2,\ C \in \mathbb{R}$ and $M > 0$.

Assumption 3. *The generator f satisfies for all $t \geq 0,\ u \in \mathbb{R}$ and $z \in \mathbb{R}^d$*

$$\frac{\partial f}{\partial u}(t,u,z) \leq I_\epsilon(t) + \epsilon|z|^2 \quad a.s. \tag{2.24}$$

for $I_\epsilon \in L^1$ and $\epsilon > 0$.

Theorem 2.2. *Comparison principle. Let (f,ξ) and $\left(\tilde{f},\tilde{\xi}\right)$ be the data of two BSDE's and suppose that:*

1. *$\xi \leq \tilde{\xi}$ a.s. and $f \leq \tilde{f}$.*
2. *For all ϵ, $M > 0$ there exists $I, I_\epsilon \in L^1,\ k \in L^2,\ C \in \mathbb{R}$ such that either f or $\tilde{f}$ satisfies Assumption 1 and Assumption 3.*

Then if (Y,Z,C) (resp. $\left(\tilde{Y},\tilde{Z},\tilde{C}\right)$) $\in H_T^\infty(\mathbb{R}) \times H_T^2\left(\mathbb{R}^d\right) \times RCI(\mathbb{R})$ is a subsolution (resp. a supersolution) of the BSDE with parameters (f,ξ) (resp. $\left(\tilde{f},\tilde{\xi}\right)$), one has

$$\forall t \geq 0, \quad Y_t \leq \tilde{Y}_t. \tag{2.25}$$

Remark 2.1. It holds true if either $f(Y,Z,C) \leq \tilde{f}(Y,Z,C)$ a.s. for all t and $\tilde{f}$, satisfy Assumption 1 and Assumption 3, or if $f\left(\tilde{Y},\tilde{Z},\tilde{C}\right) \leq \tilde{f}\left(\tilde{Y},\tilde{Z},\tilde{C}\right)$ a.s. for all t and f satisfy Assumption 1 and Assumption 3.

Proposition 2.2. *Stability of BSDEs. Let (f,ξ) be the data of a BSDE as in Theorem 2.2 and $\{(f_n,,\xi_n), n \in \mathbb{N}\}$ be a sequence of data such that:*

1. *The sequence* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *locally uniformly on* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$*, for each* $n \in \mathbb{N}$*,* $\xi_n \in L^\infty_T$ *and* $\{\xi_n\}_{n\in\mathbb{N}}$ *converges to* ξ *in* L^∞_T*.*

2. *There exists* $k : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for all* $T > 0$*,* $k \in L^\infty_1$*, and there exists* $C > 0$ *such that*

$$\forall n \in \mathbb{N},\ \forall (t,u,z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d, \quad |f_n(t,u,z)| \le k_t + C|z|^2. \tag{2.26}$$

3. *For each* n*, the BSDE with data* (f_n, ξ_n) *has a solution* (Y_n, Z_n) *such that the sequence* $\{Y_n\}_{n\in\mathbb{N}}$ *is monotonic, and there exists* $M > 0$ *such that for all* $n \in \mathbb{N}$*,* $\|Y_n\|_\infty \le M$*.*

Then if the sequence $\{f_n\}_n$ *converges to* f *locally uniformly on* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d$*, and if the sequence* $\{\xi_n\}_m$ *converges to* ξ *in* L^∞*, there exists a pair of adapted processes* (Y, Z) *such that for all* $T \in \mathbb{R}^+$*,*

$$\lim_{n\to\infty} Y_n = Y \quad \textit{uniformly on } [0,T] \tag{2.27}$$

$$\{Z_n\}_{n\in\mathbb{N}} \textit{ converges to } Z \textit{ in } H^2_T \tag{2.28}$$

and (Y, Z) *is the solution of the BSDE with data* (f, ξ)*.*

For the proof we define

$$g^n = \sup_{p\ge n} f^p, \quad h^n = \inf_{p\ge n} f^p, \tag{2.29}$$

$$\xi^{n*} = \sup_{p\ge n} \xi^p, \quad \xi^n_* = \inf_{p\ge n} \xi^p \tag{2.30}$$

and we consider the maximal solutions (Y^{n*}, Z^{n*}) of the BSDE with parameters (g^n, ξ^{n*}) and the minimal solutions (Y^n_*, Z^n_*) of the BSDE with parameters (h^n, ξ^n_*). Then from Theorem 2.2 there exists (Y^*, Z^*) such that Y^{n*} converges uniformly to Y^*, and (Y^*, Z^*) is a solution of the BSDE with data (f, ξ). Analogously, there exists (Y_*, Z_*) such that Y^n_* converges uniformly to Y_*, and (Y_*, Z_*) is a solution of the BSDE with data (f, ξ). Finally, by Theorem 2.2 we have both

$$\forall n \quad Y^n_* \le Y^n \le Y^{n*} \ \text{ and } \ Y_* = Y^* = Y; \tag{2.31}$$

therefore the sequence $\{Y_n\}_{n\in\mathbb{N}}$ converges uniformly to Y.

2.4 Numeric Methods

We present some of the algorithms used in the simulations of BSDE and useful approximations to solve BSDE with quadratic growth. First is the method of truncation. The objective is to find a sequence of functions that defines a family of BSDE and satisfies the Lipschitz condition and so we can ensure existence and uniqueness of this family of BSDE. We want to ensure existence and uniqueness to apply numerical methods such as Euler-Maruyama approximation.

2.4.1 Truncation Method

This method consist on defining a sequence of $\{f_n, \xi\}_{n\in\mathbb{N}}$ of data of BSDE such that f_n satisfies the Lipschitz condition $\forall n \in \mathbb{N}$ and $f_n \to f$, where f has quadratic growth as describes Assumption 1. For this section will show the truncation method by defining a set of functions $\{f_n(t,y,z)\}_{n\in\mathbb{N}}$ such that satisfy the Lipschitz condition on z but converge as $n \to \infty$ to a function that grows quadratically in z. For this we need to define the following functions, let $n \in \mathbb{N}$ then:

$$\hat{h}_n(x) = \begin{cases} n+1 & x > n+2, \\ \frac{-n^2+2nx-x(x-4)}{4} & n \le x \le n+2, \\ x & |x| < n, \\ \frac{n^2+2nx+x(x+4)}{4} & -(n+2) \le x \le -n, \\ -(n+1) & x < -(n+2). \end{cases} \tag{2.32}$$

Note that $\hat{h}_n(x)$ has the following properties:

- $\hat{h}_n : \mathbb{R} \to \mathbb{R}$ is a continuous differentiable function.
- $\left\{\hat{h}_n\right\}_{n\in\mathbb{N}}$ converges uniformly to the identity.
- For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ it holds that $\left|\hat{h}_n(x)\right| \le |x|$ and $\left|\hat{h}_n(x)\right| \le n+1$.
- Its derivative is absolutely uniformly bounded by 1, and converges to 1 locally uniformly.

Now we define $h_n : \mathbb{R}^d \to \mathbb{R}^d$ by

$$x \mapsto h_n(x) := \left(\hat{h}_n(x_1), \hat{h}_n(x_2), \dots, \hat{h}_n(x_d)\right). \tag{2.33}$$

Finally we define for each $n \in \mathbb{N}$, $f_n : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ by

$$(t,y,z) \to f_n(t,y,z) := f\left(t,y,h_n\left(z\right)\right),$$

where the pair (f,ξ) are somewhat similar to the ones stated in Assumptions 1 and 3.

Assumption 4. *$f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is an adapted measurable function continuously differentiable in the spatial variables. There exists a positive constant M such that for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$*

$$\left|f\left(t,y,z\right)\right| \leq M\left(1+|y|+|z|^2\right) \quad a.s., \tag{2.34}$$

$$\left|\frac{\partial}{\partial y} f\left(t,y,z\right)\right| \leq M \quad a.s., \tag{2.35}$$

$$\left|\nabla_z f\left(t,y,z\right)\right| \leq M\left(1+|z|\right) \quad a.s. \tag{2.36}$$

Assumption 5. *The random variable ξ is absolutely bounded.*

With the sequence $\{f_n\}_{n\in\mathbb{N}}$ we define a family of BSDE as follows

$$Y_s^n = \xi + \int_t^T f_n\left(s,Y_s^n,Z_s^n\right)ds - \int_t^T Z_s^n ds. \tag{2.37}$$

Lemma 2.1. *The family of BSDE $\{f_n,\xi\}_{n\in\mathbb{N}}$ has a solution and it is unique.*

Proof. We will prove that all the functions in the sequence $\{f_n\}_{n\in\mathbb{N}}$ are Lipschitz continuous in the spatial variables and because of Theorem 1.1, we prove that $\{f_n,\xi\}_{n\in\mathbb{N}}$ has a solution and it is unique.

To prove this, we will combine Assumption 5 with the properties of the sequence $\{f_n\}_{n\in\mathbb{N}}$. For the next lines let us fix $n \in \mathbb{N}$ and take $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$. Notice that in f_n only the variable z is changed by $h_n(z)$. Using equation (2.35) this means that $\left|\frac{\partial}{\partial y} f_n\left(t,y,z\right)\right| \leq M \quad a.s.$, for all (t,y,z) and hence that f_n satisfies a standard Lipschitz condition in the variable y with Lipschitz constant M.

To check that f_n also satisfies a standard Lipschitz condition in the variable z, notice that $|h_n|$ is uniformly bounded by $(n+1)$ and at the same time $\left|h_n\left(z\right)\right| \leq |z|$ for all z. According to (2.34) this translates into the existence of a positive constant

M such that for all (t, y, z)

$$|f_n(t,y,z)| = |f(t,y,h_n(z))| \tag{2.38}$$
$$\leq M\left(1+|y|+|h_n(z)|^2\right) \tag{2.39}$$
$$\leq M(1+|y|+(n+1)|z|). \tag{2.40}$$

On the other hand, since $|\nabla_z h_n(z)| \leq 1$ and by (2.36) we have for any (t, y, z)

$$|\nabla_z f_n(t,y,z)| = |\nabla_z f(t,y,h_n(z))|\,|\nabla_z h_n(z)| \tag{2.41}$$
$$\leq |\nabla_z f(t,y,h_n(z))| \tag{2.42}$$
$$\leq \tilde{M}(1+|h_n(z)|) \tag{2.43}$$
$$\leq \tilde{M}(1+(n+1)). \tag{2.44}$$

Bringing (2.40) and (2.44) together and combining them with the mean value theorem, we have for any $t \in [0, T]$, $y,\ \widetilde{y} \in \mathbb{R}$ and $z,\ \widetilde{z} \in \mathbb{R}^d$ that

$$|f_n(t,y,z) - f_n(t,\widetilde{y},\widetilde{z})| \leq \max_{(t,y,z)}\left|\frac{\partial}{\partial y} f_n(\cdot,\cdot,\cdot)\right| |y-\widetilde{y}| + \max_{(t,y,z)} |\nabla_z f_n(\cdot,\cdot,\cdot)|\,|z-\widetilde{z}| \tag{2.45}$$
$$\leq M|y-\widetilde{y}| + \tilde{M}(n+2)|z-\widetilde{z}| \tag{2.46}$$
$$\leq M_n(|y-\widetilde{y}| + |z-\widetilde{z}|), \tag{2.47}$$

with $M_n := M \vee \left[\tilde{M}(n+2)\right]$ and hence we conclude that for each $n \in \mathbb{N}$ the function f_n satisfies a standard Lipschitz condition in the spatial variables with a Lipschitz constant depending on n. □

2.4.2 Euler–Maruyama Approximation

Consider the one-dimensional BSDE

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dWt, \quad Y_T = \xi \tag{2.48}$$

where W is an m-dimensional standard Wiener process, $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is Lipschitz. Many SDE systems do not have an analytic solution, so it is necessary to solve these systems numerically: the simplest stochastic numerical approximation is the Euler–Maruyama method.

Let $\{\epsilon_i^n\}_{i=1,2,\ldots,n}$ be an i.i.d. Bernoulli sequence:

$$\mathbb{P}(\epsilon_i^n = 1) = \mathbb{P}(\epsilon_i^n = -1) = \frac{1}{2}. \tag{2.49}$$

We define

$$W_k^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \epsilon_i^n, \tag{2.50}$$

$$\Delta W_k^n = W_{k+1}^n - W_k^n = \frac{1}{\sqrt{n}} \epsilon_k^n \tag{2.51}$$

$$\Delta t_k = t_{k+1} - t_k \tag{2.52}$$

and

$$\xi^n = \Phi^n \left(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_n^n\right), \tag{2.53}$$

where Φ^n exists because of the inverse transform Theorem.

Note that W^n converges to a brownian motion W and ξ^n converges to ξ.

We will use the following notations for simplification:

$$\begin{aligned} y_k &= Y_t, \qquad z_k = Z_t \\ w_k &= W_t, \qquad f_k^n (y, z) = f (t, y, z). \ \ \forall t \in \left[\frac{k}{n}, \frac{k+1}{n}\right), \ k = 0, 1, \ldots, n. \end{aligned}$$

The Euler–Maruyama Approximation consists in solving backwadly

$$-\left(y_{k+1}^n - y_k^n\right) = f_k^n \left(y_k^n, z_k^n\right) \frac{1}{n} - z_k^n \Delta w_{k+1}^n, \ k = n-1, n-2, \ldots, 1, \tag{2.54}$$

with terminal condition $y_n^n = \xi^n$.

We can solve (2.54) setting

$$y_{k+1}^+ = \Phi^n \left(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_{n-1}^n, 1\right), \tag{2.55}$$

$$y_{k+1}^- = \Phi^n \left(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_{n-1}^n, -1\right). \tag{2.56}$$

Now, from (2.54), we can get

$$y_k^n = y_{k+1}^{+} + f_k^n\left(y_k^n, z_k^n\right)\frac{1}{n} - z_k^n\frac{1}{\sqrt{n}}, \tag{2.57}$$

$$y_k^n = y_{k+1}^{+} + f_k^n\left(y_k^n, z_k^n\right)\frac{1}{n} + z_k^n\frac{1}{\sqrt{n}},\ \ k = n-1, n-2, \ldots, 1. \tag{2.58}$$

By calculating the above equations, we get

$$\begin{aligned} z_k^n &= \frac{y_{k+1}^{(+)} - y_{k+1}^{(-)}}{2\sqrt{n}}, \\ y_k^n &= \frac{y_{k+1}^{(+)} + y_{k+1}^{(-)}}{2} + f_k^n\left(y_k^n, z_k^n\right)\frac{1}{n}. \end{aligned}$$

Finally if the generator f only depends on the parameter Z and t, we obtain the backward recursive formula:

$$\begin{aligned} z_k &= \frac{y_{k+1}^{(+)} - y_{k+1}^{(-)}}{2\sqrt{\Delta t}}, \quad \forall k = N-1, N-2, \ldots, 0; \\ y_k &= \frac{y_{k+1}^{(+)} + y_{k+1}^{(-)}}{2} + f_k^n\left(z_k^n\right)\frac{1}{n}, \quad \forall k = N-1, N-2, \ldots, 0. \end{aligned}$$

2.4.3 Simulations

For this section we will retake the example of Chapter 2

$$Y_t = \xi + \int_t^T \frac{1}{2}\left|Z_s\right|^2 ds - \int_t^T Z_s dW_s, \quad \forall t \in [0, T] \tag{2.59}$$

with $\xi \sim N(0,1)$. We know that the analytic solution for Y_t is

$$Y_t = \log\left(\mathbb{E}\left(e^{\xi}\middle| \mathcal{F}_t\right)\right) \quad \forall t \in [0, T] \tag{2.60}$$

and that $\mathbb{E}\left(e^{\xi}\right) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = \exp\left(\frac{1}{2}\right)$. Then the analytic solution of Y_0 is 0.5.

Using the Euler–Maruyama approximation with the truncation method over the variable z, we obtain the following solutions:

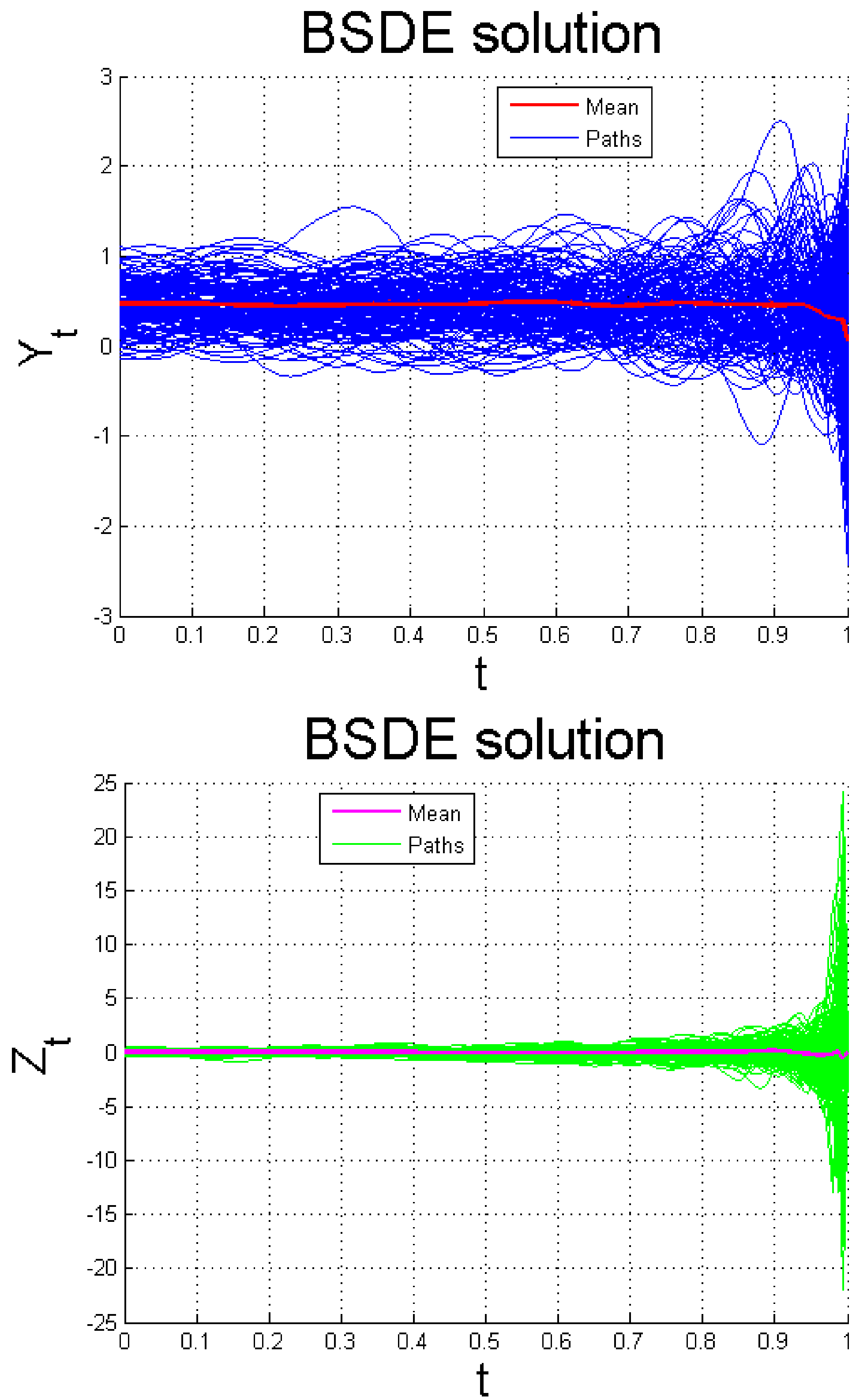

Figure 2.1: Simulation using 150 trajectories and a partition of length $\frac{1}{150}$, and $n = 15$ for the truncation.

Partitions	Trajectories	Y_0	Z_0	time elapsed (sec.)
1000	1000	0.5042	3.8419e-004	3723.603700
700	800	0.4924	4.4922e-004	1636.662872
500	500	0.4710	0.0015	562.303576
200	300	0.5513	0.0024	48.041308
100	100	0.6469	0.0439	10.562842
10	10	0.8371	-0.0502	0.919158

Table 2.1: Number of truncation $n = 25$.

Chapter 3

Optimal Control

Optimal Control theory appears in the 1950's, especially motivated by the Space Race between the Soviet Union (USSR) and the United States (US) for supremacy in space exploration. Engineers became interested in the problem of controlling a system governed by a system of differential equations. In many of the problems it was natural to want to control the system so that a given performance index would be minimized. In some aerospace problems large savings in cost could be obtained with a small improvement of optimal controls. The use of this theory became common in large number of fields.

3.1 Preliminaries, DPP and verification theorem

For this chapter we consider the following

- A filtered space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.
- We say that a feasible control $(u_t, t \in [0,T])$ is a $\mathcal{F}_t$- adapted and square integrable process valued in a compact metric space U. The set of feasible controls will be detonated by $\mathcal{U}$.
- The state process $\{X_t,\ t \in [0,T]\}$ is a $\mathcal{F}_t$- adapted and square integrable process which takes values on a Polish space S.

The laws of the control

$$dX_s = b\left(s, X_s, u_s\right) ds + \sigma\left(s, X_s, u_s\right) dW_s,\ t \leq s \leq T \quad X_t = x, \tag{3.1}$$

where:

- X_s is the state of the system at time s.
- $u(s)$ is the control applied at time s.
- W is a $n-$dimensional brownian motion.
- b and σ are deterministic functions, uniformly Lipschitz with respect to x and u with $b(s,0,0)$ and $\sigma(s,0,0)$ uniformly bounded.

Definition 3.1. At a time t and for a control u, we define an objective function as

$$J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^T L(s,X_s,u_s)\,ds + \Psi(T,X_T)\right], \tag{3.2}$$

where $\mathbb{E}_{t,x}$ denotes the conditional expectation given $X_t = x$. The deterministic functions $L(s,X_s,u_s)\,ds$ and $\Psi(T,X_T)$ correspond to the running cost associated with control u and state X and the terminal cost respectively.

Definition 3.2. The valued function is defined by

$$V(t,x) := \inf_{u\in\mathcal{U}} J(t,x,u). \tag{3.3}$$

An optimal control u^0 has the property that for any t, x,

$$V(t,x) = J\left(t,x,u^0\right). \tag{3.4}$$

Suppose that there exists an optimal control u^0, then for $t+h \leq T$

$$V(t,x) = \mathbb{E}_{t,x}\left[\int_t^T L\left(s,X_s,u_s^0\right)ds + \Psi(T,X_T)\right] \tag{3.5}$$

$$= \mathbb{E}_{t,x}\left[\int_t^{t+h} L\left(s,X_s,u_s^0\right)ds + \int_{t+h}^T L\left(s,X_s,u_s^0\right)ds + \Psi(T,X_T)\right] \tag{3.6}$$

$$= \mathbb{E}_{t,x}\left[\int_t^{t+h} L\left(s,X_s,u_s^0\right)ds + V(t+h,X_{t+h})\right], \text{ by the tower property.} \tag{3.7}$$

Now suppose that the controller uses a control u for time $r \in [t,t+h]$ and uses an optimal control u^0 after $t+h$, then

$$
\begin{aligned}
J\left(t,x,u^{1}\right) &= \mathbb{E}_{t,x}\left[\int_t^T L\left(s,X_s,u_s^1\right)ds + \Psi\left(T,X_T\right)\right] &(3.8)\\
&= \mathbb{E}_{t,x}\left[\int_t^{t+h} L\left(s,X_s,u_s\right)ds + \int_{t+h}^T L\left(s,X_s,u_s^0\right)ds + \Psi\left(T,X_T\right)\right] &(3.9)\\
&= \mathbb{E}_{t,x}\left[\int_t^{t+h} L\left(s,X_s,u_s\right)ds + V\left(t+h,X_{t+h}\right)\right], \quad \text{by the tower property,} &(3.10)
\end{aligned}
$$

where

$$
u^1(r,x) = \begin{cases} u(r,x) & \text{if } r\in[t,t+h] \\ u^0(r,x) & \text{if } r\in[t+h,T] \end{cases}. \tag{3.11}
$$

Note that $V(t,x) \leq J\left(t,x,u^1\right)$ by definition and $V(t+h,X_{t+h}) = J\left(t+h,X_{t+h},u^0\right)$, hence

$$
V(t,x) \leq \mathbb{E}_{t,x}\left[\int_t^{t+h} L\left(s,X_s,u_s\right)ds + V\left(t+h,X_{t+h}\right)\right], \tag{3.12}
$$

with equality if $u = u^0$. Now we can define the following

Definition 3.3. Dynamic Programming Principle DPP,

$$
V(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^{t+h} L\left(s,X_s,u_s\right)ds + V\left(t+h,X_{t+h}\right)\right], \tag{3.13}
$$

which means that if the controller stops at time $t+h$ then the best option is to find $V\left(t+h,X_{t+h}\right)$. So that initial control problem at time t associated with terminal value T is equivalent to the problem of minimizing the criteria associated with terminal value $t+h$ and terminal cost $V\left(t+h,X_{t+h}\right)$.

A natural question is what happens when $h\downarrow 0$? Is there an equivalent problem? For answering those questions we use (3.12),

$$
0 \leq \mathbb{E}_{t,x}\left[\int_t^{t+h} L\left(s,X_s,u_s\right)ds + V\left(t+h,X_{t+h}\right) - V\left(t,X_t\right)\right], \tag{3.14}
$$

then by Itô's formula to $V(s,X_s)$ for all $s\in[t,t+h]$

$$
\begin{aligned}
V\left(t+h,X_{t+h}\right) - V\left(t,X_t\right) &= \int_t^{t+h} \partial_t V\left(s,X_s\right)ds + \int_t^{t+h} \partial_x V(s,x)\,dX_s + &(3.15)\\
&+ \frac{1}{2}\int_t^{t+h} \partial^2_{x,x} V(s,x)\,d\left\langle X\right\rangle_s. &(3.16)
\end{aligned}
$$

Now from (3.14) and (3.15) we deduce

$$0 \leq \mathbb{E}_{t,x} \left\{ \int_t^{t+h} \left[L(s, X_s, u_s) + \partial_t V(s, X_s) + \mathcal{L}^u V(s, X_s) \right] ds \right\}, \tag{3.17}$$

where $\mathcal{L}^u$ is an operator of X, defined as

$$\mathcal{L}^u_{t,x} := \frac{1}{2} \mathrm{Tr}\left(\sigma\sigma'\right)(t, x, u)\, \partial^2_{x,x} + b(t, x, u) \cdot \partial_x. \tag{3.18}$$

Dividing by h and letting $h \downarrow 0$ from (3.17), by the mean value theorem for integrals

$$0 \leq L(t, X_t, u_t) + \partial_t V(t, X_t) + \mathcal{L}^u V(t, X_t). \tag{3.19}$$

The equality holds if $u = u^0$, in this case we will have the Hamilton-Jacobi-Bellman equation.

$$0 = \partial_t V(t, X_t) + \inf_{u \in \mathcal{U}} \left[L(t, X_t, u_t) + \mathcal{L}^u V(t, X_t) \right]. \tag{3.20}$$

This equation leads to the following theorem

Theorem 3.1. *Verification theorem. If $V(t, x)$ is a solution of (3.20) with $V(T, x) = \Psi(T, x)$ and $u^0 \in \mathcal{U}$ such that achieves the minimum in (3.20) then*

$$V(t, x) = J\left(t, x, u^0\right) = \inf_{u \in \mathcal{U}} J(t, x, u), \tag{3.21}$$

i.e., $V(t, x)$ is the value function and u^0 the optimal control.

3.2 Stochastic Control Problem

The laws of controlled processes belong to a family of equivalent measures whose densities are given by:

$$d\beta^u_t = \beta^u_t \left[d(t, u_t)\, dt + n(t, u_t)^* \, dW_t \right], \tag{3.22}$$

where $d(t, u_t)$ and $n(t, u_t)$ are uniformly bounded predictable processes. It has an explicit solution which is

$$\beta^u_t = \underbrace{\exp\left\{ \int_0^t d(t, u_t)\, dt \right\}}_{D^u_t} \underbrace{\exp\left\{ \int_0^t n(t, u_t)\, dW_t - \frac{1}{2} \int_0^t n^2(t, u_t)\, dt \right\}}_{L^u_t}. \tag{3.23}$$

Note that L_t^u is an exponential martingale, $L_0^u = 1$, $\mathbb{E}_{\mathbb{P}}\left(L_t^u\right) = 1$. So we can define the equivalent probability measure $\mathbb{Q}$ as follows

$$\mathbb{Q}\left(A\right) = \int_A \underbrace{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}}_{L_t} d\mathbb{P}.$$

Then the controller acts are given by

$$dL_t^u = L_t^u n\left(t, u_t\right)^* dW_t, \tag{3.24}$$

and the controlled discount factor by

$$dD_t^u = D_t^u d\left(t, u_t\right)^* dt. \tag{3.25}$$

The problem is to minimize the objective function $J(u)$ over all feasible control u, where

$$J\left(u\right) = \mathbb{E}\left[\int_0^T \beta_s \underbrace{f\left(s, u_s, X_s\right)}_{\text{running cost}} ds + \beta_T \underbrace{g\left(X_T\right)}_{\text{terminal cost}}\right].$$

Using the equivalent probability measure $\mathbb{Q}^u$, the objective function can be rewritten

$$J\left(u\right) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T D_s \underbrace{f\left(s, u_s, X_s\right)}_{\text{running cost}} ds + D_T \underbrace{g\left(X_T\right)}_{\text{terminal cost}}\right];$$

note that, $J(u) = Y_0^u$, where (Y^u, Z^u) is the BSDE with data (f^u, ξ^u), where

$$f^u\left(t, y, z\right) = f(t, u_t) + d\left(t, u_t\right) y + n\left(t, u_t\right)^* z \tag{3.26}$$

$$\xi^u = g\left(X_T\right). \tag{3.27}$$

The process β^u corresponds to the adjoint process associated with (Y^u, Z^u) and

$$Y_t^u = \mathbb{E}\left[\int_t^T \beta_{t,s}^u f(t, u_t) ds + \beta_{t,T}^u \xi^u \middle| \mathcal{F}_t\right]. \tag{3.28}$$

For each control u, Y_t^u corresponds to the objective function at time t. Let $\bar{Y}_t$ be the value function at time t i.e.

$$\overline{Y_t} = \text{ess} \inf_{u \in \mathcal{U}} Y_t^u, \quad 0 \leq t \leq T. \tag{3.29}$$

In this case the DPP is

$$\overline{Y_t} = \text{ess} \inf_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^{t+h} \beta_{t,s}^u f(t, u_t) ds + \beta_{t,t+h}^u \overline{Y}_{t+h} \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq t+h \leq T. \tag{3.30}$$

3.3 BSDE and Optimal Control

Let S be a system whose evolution is described by a $\mathbb{R}^d$-valued stochastic process $(X_t)_{t \leq T}$ solution of the following standard SDE:

$$dX_t = \sigma(t, X_t) \, dW_t, \quad 0 < t \leq T; \; X_0 = x \in \mathbb{R}^d. \tag{3.31}$$

The matrix σ is Lipschitz continuous with respect to x, is invertible and its inverse is bounded. A controller intervenes on this system via an adapted stochastic process $u := (u_t)_{t \in [0,T]}$ taking its values in a compact metric space U and with admissible set of controls $\mathcal{U}$. When the controller acts with u, the dynamics of the controlled system is the same as that of X under the probability measure $\mathbb{P}^u$ whose density with respect to $\mathbb{P}$ is given by:

$$\frac{d\mathbb{P}^u}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \sigma^{-1}(s, X_s) \, f(s, X_s, u_s) \, dW_s \right), \tag{3.32}$$

where

$$\mathcal{E}(M_t) := \exp\left(M_t - \frac{1}{2} \langle M \rangle_t \right), \tag{3.33}$$

is the exponential local martingale associated with the martingale M and the function f is assumed to be measurable and bounded.

Under the new probability measure $\mathbb{P}^u$, the process X is a solution of a FSDE, now driven by the $\mathbb{P}^u$-BM W^u, so that $\forall t \in [0, T]$:

$$dX_t = f(t, X_t, u_t) \, dt + \sigma(t, X_t) \, dW_t^u, \quad 0 < t \leq T, \; X_0 = x \tag{3.34}$$

$$\text{where} \quad dW_t^u = dW_t - \sigma^{-1}(t, X_t) \, f(t, X_t, u_t) \, dt. \tag{3.35}$$

Those considerations imply that the action of the controller raises a drift in the dynamics of the system. On the other hand, the control action is not free and generates a profit for the agent, denoted by $J(u)$ and equal to:

$$J(u) := \mathbb{E}^u \left[\int_0^T h(s, X_s, u_s)\, ds + \Psi(X_T) \right]. \tag{3.36}$$

The problem is now to find $u^* \in \mathcal{U}$ such that $J(u^*) \geq J(u)$ for any $u \in \mathcal{U}$. We assume the following strong properties: f and h are bounded, continuous with respect to u. The terminal cost function Ψ is also assumed to be bounded.

Theorem 3.2. *Under the previous assumptions, for any admissible control $u \in \mathcal{U}$, the hamiltonian processes $H(t, x, z, u)$ and the maximal hamiltonian process $H^*(t, x, z)$:*

$$H(t, x, z, u) := z\sigma^{-1}(t, x) f(t, x, u) + h(t, x, u) \tag{3.37}$$
$$H^*(t, x, z) := \sup_{u \in \mathcal{U}} H(t, x, z, u), \tag{3.38}$$

define a family of BSDEs with terminal condition $\Psi(X_T)$, and linear or convex coefficients H, H^* that lives in H_T^2 for $(y, z) = (0, 0)$ and satisfies the uniformly Lipschitz condition with respect to (Y, Z). The associated solutions are denoted by (Y^u, Z^u) and (Y^*, Z^*).

Moreover, there exists a measurable control process $u_t^* = u^*(t, x, Z_t^*)$ such that at any time $t \in [0, T]$ and for any $z \in \mathbb{R}^m$, $H^*(t, x, z) = H(t, x, z, u^*(t, x, z))$. The process $u^*(t, x, z)$ is an optimal control process since at any time $t \leq T$:

$$Y_t^* = Y_t^{u^*} = ess \sup_{u \in \mathcal{U}} Y_t^{u_t}; \quad in\ particular \;\; Y_o^* = \sup_{u \in \mathcal{U}} J(u) = J^*. \tag{3.39}$$

Proof. Since $\sigma^{-1}(t, X_t) f(t, X_t, u_t)$, $h(t, X_t, u_t)$ and $\Psi(X_T)$ are uniformly bounded, $H(t, x, z, u)$ is a linear generator of a BSDE that satisfies the hypothesis of Proposition 1.2 which states that the BSDE with data $(H(t, x, z, u), \Psi(X_T))$ has a unique $\mathbb{P}-$solution (Y^u, Z^u)

$$-dY_t^u = H(t, x, Z_t^u, u_t)\, dt - Z_t^u dW_t; \quad Y_T = \Psi(X_T), \tag{3.40}$$

such that

$$
\begin{aligned}
Y_o^u =& \Psi\left(X_T\right)+\int_t^T H\left(s, X_s, Z_s^u, u_s\right) ds-\int_t^T Z_s^u dW_s && (3.41)\\
=& \Psi\left(X_T\right)+\int_t^T \left(Z_s^u \sigma^{-1}\left(s, X_s\right) f\left(s, X_s, u_s\right)+h\left(s, X_s, u_s\right)\right) ds+ && (3.42)\\
&-\int_t^T Z_s^u\left[dW_t^u+\sigma^{-1}\left(t, X_t\right) f\left(t, X_t, u_t\right) dt\right] && (3.43)\\
=& \Psi\left(X_T\right)+\int_t^T h\left(s, X_s, u_s\right) ds-\int_t^T Z_s^u dW_s^u, && (3.44)
\end{aligned}
$$

taking the expected value in (3.44) we obtain

$$
\begin{aligned}
Y_o^u &= \mathbb{E}_{\mathbb{P}^u}\left[\Psi\left(X_T\right)+\int_t^T h\left(s, X_s, u_s\right) ds\right] && (3.45)\\
&= J(u). && (3.46)
\end{aligned}
$$

The comparison Theorem 1.2 suggests taking $H^*(t, x, z)$ as the generator of the BSDE associated to the solution (Y^*, Z^*) as a supremum of uniformly Lipschitz affine coefficient, since

$$
\begin{aligned}
\left|\sup_{u \in U} H(t, x, z, u)-\sup_{u \in U} H(t, x, \tilde{z}, u)\right| &\leq \sup_{u \in U}\left|H(t, x, z, u)-H(t, x, \tilde{z}, u)\right| \\
&\leq k|z-\tilde{z}|;
\end{aligned}
$$

moreover, $H^*(t, X_t, 0)=\sup_{u \in \mathcal{U}} h(t, X_t, u_t)$.

For the measurability of $H^*(t, x, z)$ given that we take a supremum over an uncountable set. For the existence look at Appendix A.1. Therefore, for Theorem 1.1 there is a unique $\mathbb{P}-$solution (Y^*, Z^*). □

3.4 An example in finance with qgBSDE

We finish this thesis with an example of pricing and hedging of derivatives. This example envolves all the topics discussed throughout this thesis. This example uses exponential utility and it can be seen as a stochastic optimal control problem that can be solved via BSDE with quadratic growth.

We calculate exponential utility-based indifference prices, and corresponding derivative hedges using the fact that they can be represented in terms of solutions of BSDE with quadratic growth.

3.4.1 Pricing and hedging of derivatives

Let $\left(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a filtered space and W be a d-dimensional brownian motion. Suppose that a derivative with maturity T is based on an $\mathbb{R}^m$-dimensional non-tradable index (think of a stock, temperature or loss index) with dynamics

$$dR_t = b\left(t, R_t\right) dt + \rho\left(t, R_t\right) dW_t, \quad R_0 = r \in \mathbb{R}^m, \quad t \in [0, T], \tag{3.47}$$

where $b : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\rho : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are measurable continuous functions. Throughout we assume that ρ and b satisfy that there exists a positive constant C such that for all $t \in [0, T]$ and $r, \widetilde{r} \in \mathbb{R}^m$

$$\left|b\left(t, r\right) - b\left(t, \widetilde{r}\right)\right| + \left|\rho\left(t, r\right) - \rho\left(t, \widetilde{r}\right)\right| \leq C\left|r - \widetilde{r}\right|, \tag{3.48}$$

$$\left|b\left(t, r\right)\right| + \left|\rho\left(t, r\right)\right| \leq C\left(1 + |r|\right). \tag{3.49}$$

We consider a derivative of the form $F\left(R_T\right)$, where $F : \mathbb{R}^m \to \mathbb{R}$ is a uniformly bounded continuous function. Note that at time $t \in [0, T]$, the payoff of $F\left(R_T\right)$, conditioned on $R_t = r \in \mathbb{R}^m$, is given by $F\left(R_T^{t,r}\right)$, where $R_s^{t,r}$ with $s \in [t, T]$ is the solution of the SDE

$$R_s^{t,r} = r + \int_t^s b\left(u, R_u^{t,r}\right) du + \int_t^s \rho\left(u, R_u^{t,r}\right) dW_u, \quad s \in [t, T], \quad r \in \mathbb{R}^m. \tag{3.50}$$

Our correlated financial market consists of k risky assets and one riskless asset. We use the riskless asset as the numeraire and suppose that the prices of the risky assets in units of the numeraire evolve according to the SDE

$$\frac{dS_t^i}{S_t^i} = \alpha_i\left(t, R_t\right) dt + \beta_i(t, R_t) dW_t, \quad i \in \{1, 2, \ldots, k\}, \quad t \in [0, T], \tag{3.51}$$

where α_i is the i-th component of a measurable and vector-valued map $\alpha : [0, T] \times \mathbb{R}^m \to \mathbb{R}^k$ and β_i is the i-th row of a measurable and matrix-valued map $\beta : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{k\times d}$. The correlation between the index and the tradable assets is determined by the matrices ρ and β. In order to exclude arbitrage opportunities in the financial market we assume $d \geq k$.

For technical reasons we suppose that:

Assumption 6. *The functions α and β are measurable continuous functions. Throughout we assume that α and β satisfy that there exists a positive constant D such that for all $t \in [0,T]$ and $r, \widetilde{r} \in \mathbb{R}^m$,*

$$|\alpha(t,r) - \alpha(t,\widetilde{r})| + |\beta(t,r) - \beta(t,\widetilde{r})| \leq D|r - \widetilde{r}|, \tag{3.52}$$

$$|\alpha(t,r)| + |\beta(t,r)| \leq D(1 + |r|). \tag{3.53}$$

Furthermore, there exist two constants $0 < \epsilon < K$ such that $\epsilon I_k \leq (\beta\beta^*)(t,r) \leq K I_k$ for all $(t,r) \in [0,T] \times \mathbb{R}^m$, where β^* is the transpose of β, and I_k is the k-dimensional unit matrix.

Let U be the exponential utility function with risk aversion coefficient $\eta > 0$, i.e.

$$U(x) = -e^{-\eta x}, \quad x \in \mathbb{R}. \tag{3.54}$$

In what follows let $(t,r) \in [0,T] \times \mathbb{R}^m$. By an investment strategy we mean any predictable process $\lambda = (\lambda^i)_{i \leq k}$ with values in $\mathbb{R}^k$ such that the integral process $\int_0^t \lambda_u^i \frac{dS_u^i}{S_u^i}$ is well-defined for all $i \in \{1, \ldots, k\}$. We interpret λ^i as the portfolio fraction invested in the i-th asset. Investing according to a strategy λ in the time interval $[t,s]$ with $0 \leq t \leq s \leq T$ leads to a total gain due to trading described by

$$G_s^{\lambda,t} = \sum_{i=1}^k \int_t^s \lambda_u^i \frac{dS_u^i}{S_u^i}. \tag{3.55}$$

We will denote by $G_s^{\lambda,t,r}$ the gain conditional on $R_t = r$, with $(t,r) \in [0,T] \times \mathbb{R}^m$, $s \in [t,T]$ and investment strategy λ.

As one can see, for a trading strategy the wealth process conditioned on $R_t = r$ is given by

$$G_s^{\lambda,t,r} = \sum_{i=1}^k \int_t^s \lambda_u^i \left[\alpha_i\left(u, R_u^{t,r}\right) du + \beta_i\left(u, R_u^{t,r}\right) dW_u\right], \quad 0 \leq t \leq s \leq T, \; r \in \mathbb{R}^m. \tag{3.56}$$

Note that the wealth process does not depend on the value of the correlated price process. This feature of the model will later imply the indifference price at time t to depend only on the value of the index process at a given time t.

Definition 3.4. (Admissible strategy). Let $(t,r) \in [0,T] \times \mathbb{R}^m$. Define $\mathcal{A}^{t,r}$ to be the set of all strategies such that

$$\mathbb{E}\left[\int_t^T \left|\lambda_s \beta\left(s, R_s^{t,r}\right)\right|^2 ds\right] < \infty \tag{3.57}$$

and the family

$$\left\{\exp\left(-\eta G_{\tau}^{\lambda,t,r}\right) : \tau \text{ is a stopping time with values in } [0.T]\right\} \tag{3.58}$$

is uniformly integrable. Then we say that a strategy λ is admissible if $\lambda \in \mathcal{A}^{t,r}$.

The maximal expected utility at time T for an agent that does not have the derivative F on his portfolio, conditioned on his actual wealth v at time t and the level of the index $R_t = r$, is defined for $(t, v, r) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$ by

$$V^0(t, v, r) = \sup_{\lambda \in \mathcal{A}^{t,r}} \left\{\mathbb{E}\left[U\left(v + G_T^{\lambda,t,r}\right)\right]\right\}. \tag{3.59}$$

One can show that there exists a strategy π, called optimal strategy, such that

$$\mathbb{E}\left[U\left(v + G_T^{\pi,t,r}\right)\right] = V^0(t, v, r), \tag{3.60}$$

for $(t, v, r) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$. The convexity of the utility function implies that π is a.s. unique on $[t, T]$. It can be proved that $\pi \in \mathcal{A}^{t,r}$.

Suppose an investor is endowed with a derivative $F(R_T)$ and is keeping it in his portfolio until maturity T. Then his maximal expected utility is given for $(t, v, r) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$ by

$$V^F(t, v, r) = \sup_{\lambda \in \mathcal{A}^{t,r}} \left\{\mathbb{E}\left[U\left(v + G_T^{\lambda,t,s} + F\left(R_T^{t,r}\right)\right)\right]\right\}, \tag{3.61}$$

also in this case there exists an optimal strategy, denoted by $\hat{\pi}$, that satisfies

$$\mathbb{E}\left[U\left(v + G_T^{\hat{\pi},t,r} + F\left(R_T^{t,r}\right)\right)\right] = V^F(t, v, r), \tag{3.62}$$

for $(t, v, r) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$.

The presence of the derivative $F(R_T)$ in the agent's portfolio leads to a change in the optimal strategy from π to $\hat{\pi}$. The difference

$$\Delta = \hat{\pi} - \pi \tag{3.63}$$

is needed in order to hedge, at least partially, the risk associated with the derivative in the portfolio. We therefore call Δ the derivative hedge.

It can be shown that $\forall (t,r) \in [0,T] \times \mathbb{R}^m$ that there exists a real number $p(t,r)$ such that for all $v \in \mathbb{R}$

$$V^F(t, v - p(t,r), r) = V^0(t,v,r). \tag{3.64}$$

If an investor has to pay $p(t,r)$ for the derivative $F\left(R_T^{t,r}\right)$, then he is indifferent between buying and not buying the derivative. Therefore, the number $p(t,r)$ is called indifference price at time t and level r of the derivative F. It turns out that the derivative hedge is closely related to the indifference price of the derivative. The derivative either diversifies or amplifies the risk exposure of the portfolio. The difference between $\hat{\pi}$ and π measures the diversifying impact of F. The price sensitivity, i.e. the derivative of p relative to the index evolution, is also a measure of the diversification of F.

The problem of finding the optimal strategies $\hat{\pi}$ and π is a stochastic control problem. One can tackle it by solving the related Hamilton-Jacobi- Bellman equation, using a verification theorem and proving a uniqueness result. For this example we are going to use a stochastic approach, using the fact that the stochastic control problem can be solved by finding the solution of a backward stochastic differential equation (BSDE).

3.4.2 The BSDE

In order to find the value function

$$V^F(t,r) = \sup_{\lambda \in \mathcal{A}^{t,r}} \left\{ \mathbb{E}\left[-e^{-\eta\left(G_T^{\lambda,t,r} + F\left(R_T^{t,r}\right)\right)} \right] \right\}, \tag{3.65}$$

and an optimal strategy π, we construct a family of stochastic processes $R^{(\lambda)}$ with the following properties:

- $R_T^{(\lambda)} = -e^{-\eta\left(G_T^{\lambda,t,r} + F\left(R_T^{t,r}\right)\right)}$, for all $\lambda \in \mathcal{A}^{t,r}$.
- $R_0^{(\lambda)} = R_0$ is constant for all $\lambda \in \mathcal{A}^{t,r}$.
- $R^{(\lambda)}$ is a supermartingale for all $\lambda \in \mathcal{A}^{t,r}$ and there exists a $\pi \in \mathcal{A}^{t,r}$ such that $R^{(\pi)}$ is a martingale.

To construct this family, we set

$$R_t^{(\lambda)} = -e^{-\eta\left(G_t^{\lambda,t,r} + Y_t\right)} \tag{3.66}$$

where (Y, Z) is a solution of the BSDE

$$Y_t = v + F\left(R_T^{t,r}\right) - \int_t^T f\left(s, Z_s\right) ds - \int_t^T Z_s^* dW_s. \tag{3.67}$$

Now, by the Doob-Meyer descomposition exists a (local) martingale $M^{(\lambda)}$ and a (not strictly) decreasing process $A^{(\lambda)}$ that is constant for some $\lambda \in \mathcal{A}^{t,r}$.

Hu, Imkeller and Muller proved in [15] that the BSDE satisfies (3.67) has as generator $f : [0,T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ defined as follows

$$f\left(t, r, z\right) := z\theta\left(t, r\right) + \frac{1}{2\eta}\left|\theta\left(t, r\right)\right|^2 - \frac{\eta}{2}\mathrm{dist}^2\left(z + \frac{1}{\eta}\theta\left(t, r\right), C\left(t, r\right)\right) \tag{3.68}$$

where $\theta\left(t, r\right) := \left(\beta^*\left(\beta\beta^*\right)^{-1}\alpha\right)\left(t, r\right)$ and $C(t, r) := \left\{x\beta\left(t, r\right) : \ x \in \mathbb{R}^k\right\}$ be the imposing restrictions on the investor when trading in the market and is assumed to be a closed set. The distance of a vector $z \in \mathbb{R}^d$ to the closed and convex set $C(t, r)$ is defined as $\mathrm{dist}(z, C(t, r)) = \min\{|z - u| : u \in C(t, r)\}$. Because $C(t, r)$ is a linear subspace of R^d we can write for any element $z \in \mathbb{R}^d$, $\mathrm{dist}^2\left(z, C\left(t, r\right)\right) = \left|z - \Pi_{C(t,r)}\left[z\right]\right|^2$ where $\Pi_{C(t,r)}\left[z\right]$ is defined to be the projection operator of elements in $\mathbb{R}^d$ onto the subspace $C(t, r)$, i.e., $\Pi_{C(t,r)}\left[z\right] = z\left(\beta^*\left(\beta\beta^*\right)^{-1}\beta\right)\left(t, r\right)$ $\forall z \in \mathbb{R}^d$ and $(t, r) \in [0, T] \times \mathbb{R}^m$.

For the generator f we are going to check that it satisfies the Assumption 4. Under Assumption 6 the mappings β and α are uniformly bounded, hence there exists a positive constant M such that for all $(t, v, r) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$

$$\left|f\left(t, r, z\right)\right| \leq M\left(1 + |z|^2\right). \tag{3.69}$$

For the derivative $\nabla_z f$ we use the linearity of the projection operator

$$\nabla_z f\left(t, v, r\right) = \nabla_z\left(z\theta\left(t, r\right) + \frac{1}{2\eta}\left|\theta\left(t, r\right)\right|^2 - \frac{\eta}{2}\left|z + \frac{1}{\eta}\theta\left(t, r\right) - \Pi_{C(t,r)}\left[z + \frac{1}{\eta}\theta\left(t, r\right)\right]\right|^2\right) \tag{3.70}$$

$$= \nabla_z\left(z\theta\left(t, r\right) + \frac{1}{2\eta}\left|\theta\left(t, r\right)\right|^2 - \frac{\eta}{2}\left|\left(z + \frac{1}{\eta}\theta\left(t, r\right)\right)\left(I_d - \Pi_{C(t,r)}\left[I_d\right]\right)\right|^2\right) \tag{3.71}$$

$$= \theta\left(t, r\right) - \eta\left(z + \frac{1}{\eta}\theta\left(t, r\right)\right)\left(I_d - \Pi_{C(t,r)}\left[I_d\right]\right)^2, \tag{3.72}$$

hence there exists a positive constant $\widetilde{M}$ such that for all $(t,v,r) \in [0,T] \times \mathbb{R} \times \mathbb{R}^m$

$$|\nabla_z f(t,r,z)| \leq \widetilde{M}(1+|z|). \tag{3.73}$$

Combining the above inequality with the mean value theorem we obtain that a positive constant $\hat{M}$ exists such that for all $t \in [0,T]$, $r \in \mathbb{R}^m$ and $z, \widetilde{z} \in \mathbb{R}^d$

$$|f(t,r,z) - f(t,r,\widetilde{z})| \leq \hat{M}(1+|z|+|\widetilde{z}|)|z-\widetilde{z}|. \tag{3.74}$$

Also, there exists a positive Lipschitz constant $\breve{M}$ such that for any $t \in [0,T]$, $z \in \mathbb{R}^d$ and $r, \widetilde{r} \in \mathbb{R}^m$,

$$|f(t,r,z) - f(t,\widetilde{r},z)| \leq \breve{M}\left(1+|z|^2\right)|r-\widetilde{r}|. \tag{3.75}$$

We can now conclude by stating that the driver f fulfills the conditions of Assumption 4.

3.4.3 Solving the Optimal Control via BSDE

Consider the BSDE with data $\left(F\left(R_T^{t,r}\right), f\right)$, then from Theorem 2.1 and Theorem 2.2 there exists a unique solution $\left(\hat{Y}^{t,r}, \hat{Z}^{t,r}\right)$. The value function of the stochastic control problem is equal to the utility of the starting point of the BSDE, i.e.

$$V^F(t,v,r) = -e^{-\eta\left(v-\hat{Y}^{t,r}\right)}, \tag{3.76}$$

see Theorem 7 in [15].

Moreover we can reconstruct the optimal strategy $\hat{\pi}$ from $\hat{Z}$,

$$\hat{\pi}_s \beta\left(s, R_s^{t,r}\right) = \Pi_{C\left(s,R_s^{t,r}\right)}\left[\hat{Z}_s^{t,r} + \frac{1}{\eta}\theta\left(s, R_s^{t,r}\right)\right], \quad s \in [t,T]. \tag{3.77}$$

Analogously, let $(Y^{t,r}, Z^{t,r})$ be the solution of the BSDE with data $(0, f)$. Which represents a stochastic control problem as above, just without the derivative as terminal condition i.e. the derivative is not in the portfolio. In this case the maximal expected utility verifies

$$V^0(t,v,r) = -e^{-\eta\left(v-Y^{t,r}\right)}, \tag{3.78}$$

and the optimal strategy π satisfies

$$\pi_s \beta\left(s, R_s^{t,r}\right) = \Pi_{C\left(s,R_s^{t,r}\right)}\left[Z_s^{t,r} + \frac{1}{\eta}\theta\left(s, R_s^{t,r}\right)\right], \quad s \in [t,T]. \tag{3.79}$$

Since $\Pi_{C\left(s,R_s^{t,r}\right)}$ is a linear operator, the derivative hedge is given by the explicit formula

$$\Delta_s\beta\left(s,R_s^{t,r}\right)=\Pi_{C\left(s,R_s^{t,r}\right)}\left[\hat{Z}_s^{t,r}-Z_s^{t,r}\right],\quad s\in[t,T]. \tag{3.80}$$

3.4.4 The simulation

For this example we assume the following:

- The dimensions $m=d=1$.
- The risk aversion coefficient $\eta=5$, $\beta=2$ and $\alpha=1$.
- Number of iterations 70.
- Number of partitions 120.
- Number of truncation 25,
- Initial capital $v=\frac{1}{2}$.
- We consider a derivative of the form $F\left(R_T^{0,\frac{1}{2}}\right)=R_T^{0,\frac{1}{2}}$, where $R_s^{0,\frac{1}{2}}$ is the solution of the SDE

$$R_t^{0,\frac{1}{2}}=\frac{1}{2}+\int_0^t b\left(s,R_s^{0,\frac{1}{2}}\right)ds+\int_0^t \rho\left(s,R_s^{0,\frac{1}{2}}\right)dW_s,\quad t\in[0,T], \tag{3.81}$$

  where $b\left(s,R_s^{0,\frac{1}{2}}\right)=\frac{1}{10}R_t^{0,\frac{1}{2}}$ and $\rho\left(s,R_s^{0,\frac{1}{2}}\right)=3R_t^{0,\frac{1}{2}}$.

First, we simulate the prices of the derivative $F\left(R_T^{0,\frac{1}{2}}\right)$, that the follows the dynamic (3.81), using Precios.m (A.3.1).

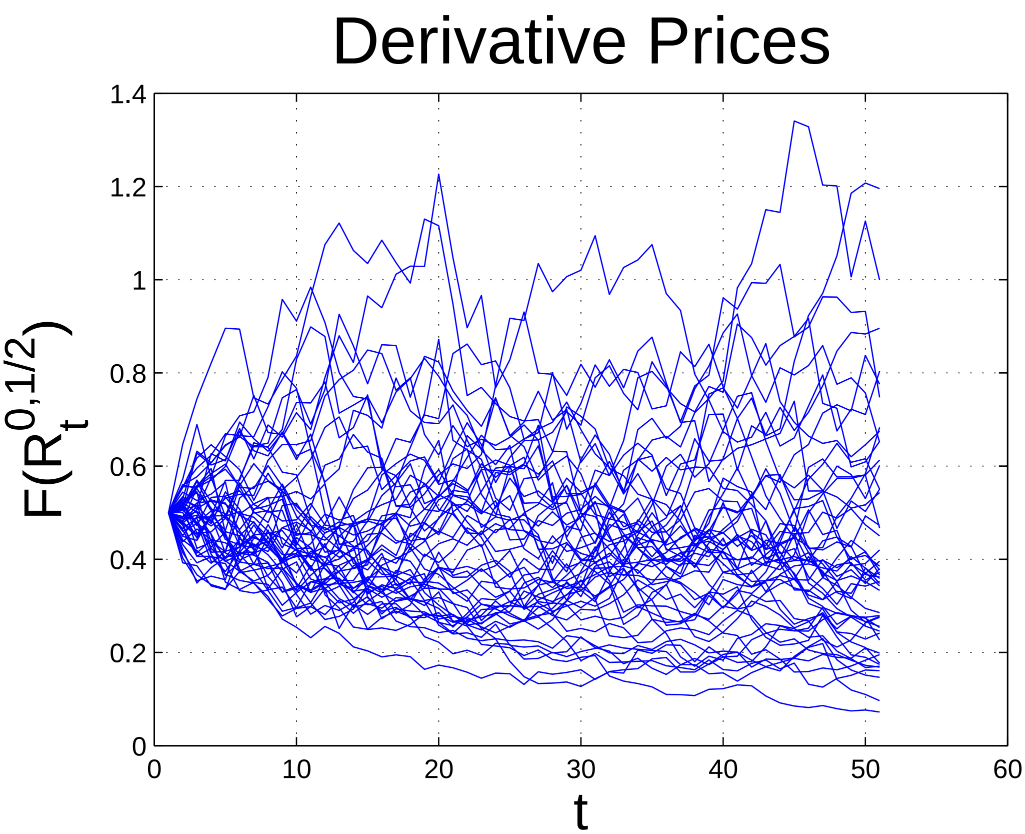

Figure 3.1:

Then, we solve the BSDE with terminal condition ξ given by the terminal value $F\left(R_T^{0,\frac{1}{2}}\right)$ from Precios.m (A.3.1) and generator f described as (3.68). Also, we calculate de BSDE with data $(0,\xi)$ using BSDE.m (A.3.1).

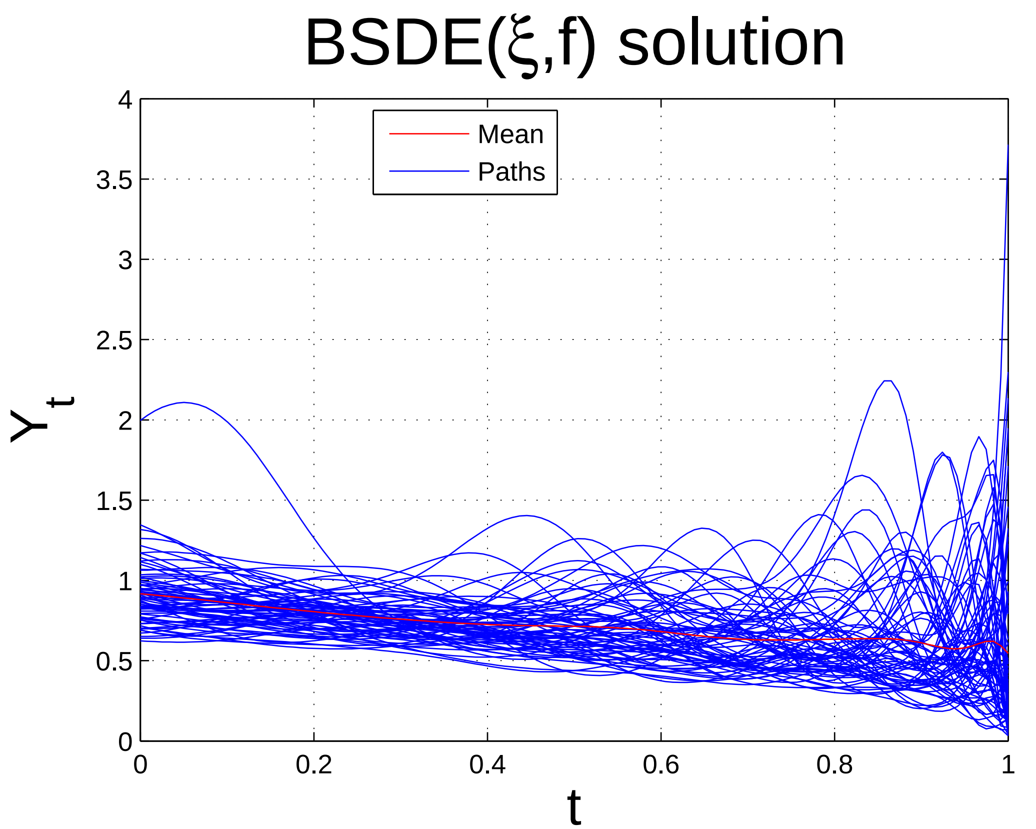

Figure 3.2:

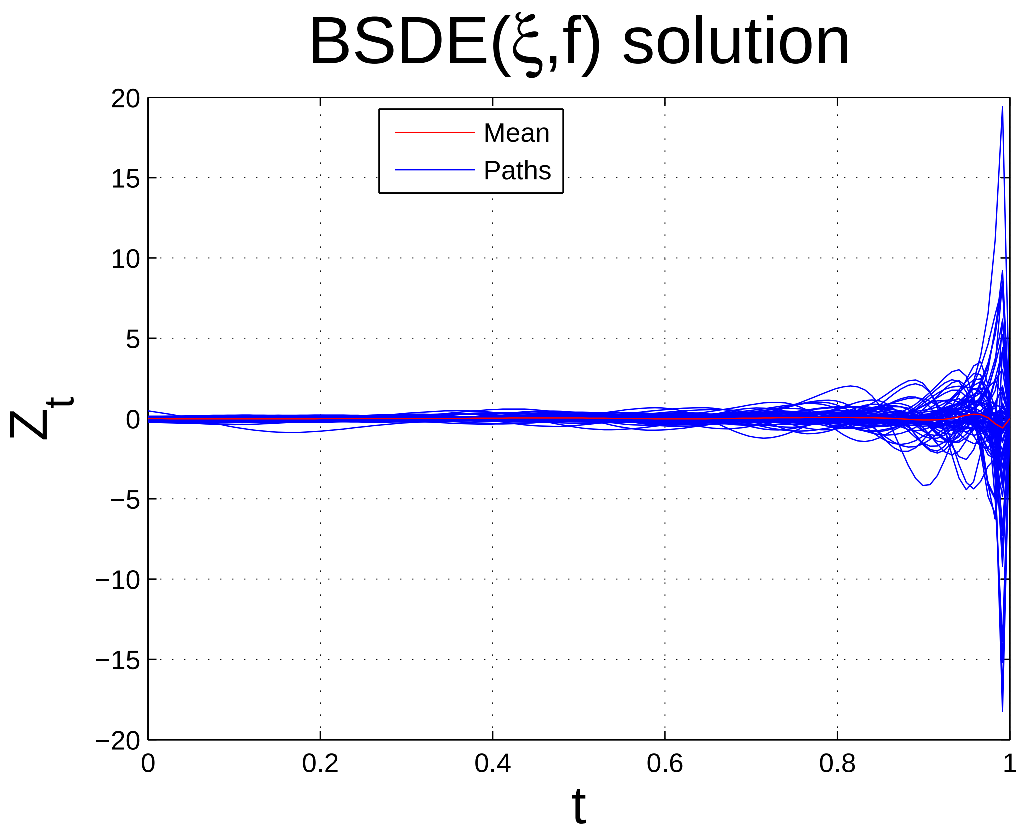
BSDE(ξ,f) solution
Mean
Paths
Z_t
t
20
15
10
5
0
−5
−10
−15
−20
0
0.2
0.4
0.6
0.8
1

Figure 3.3:

From (3.63) we calculate the hedge with the solutios of the BSDEs.

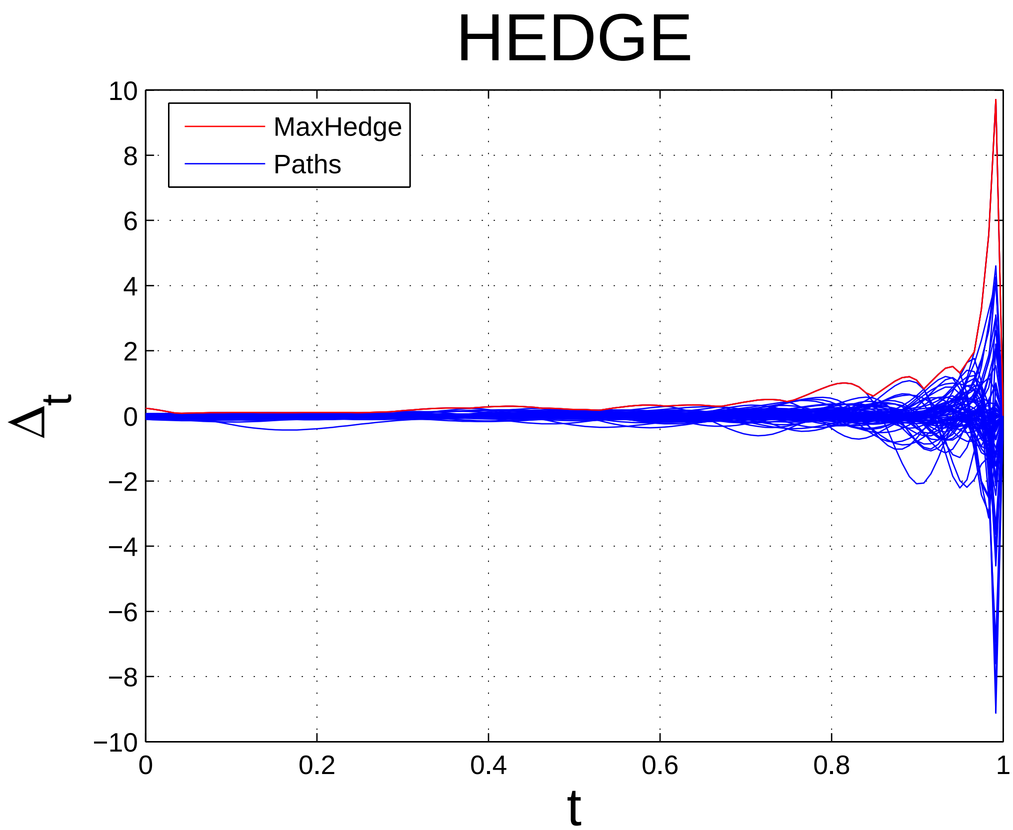

Figure 3.4:

Also, we estimate the value function from (3.76).

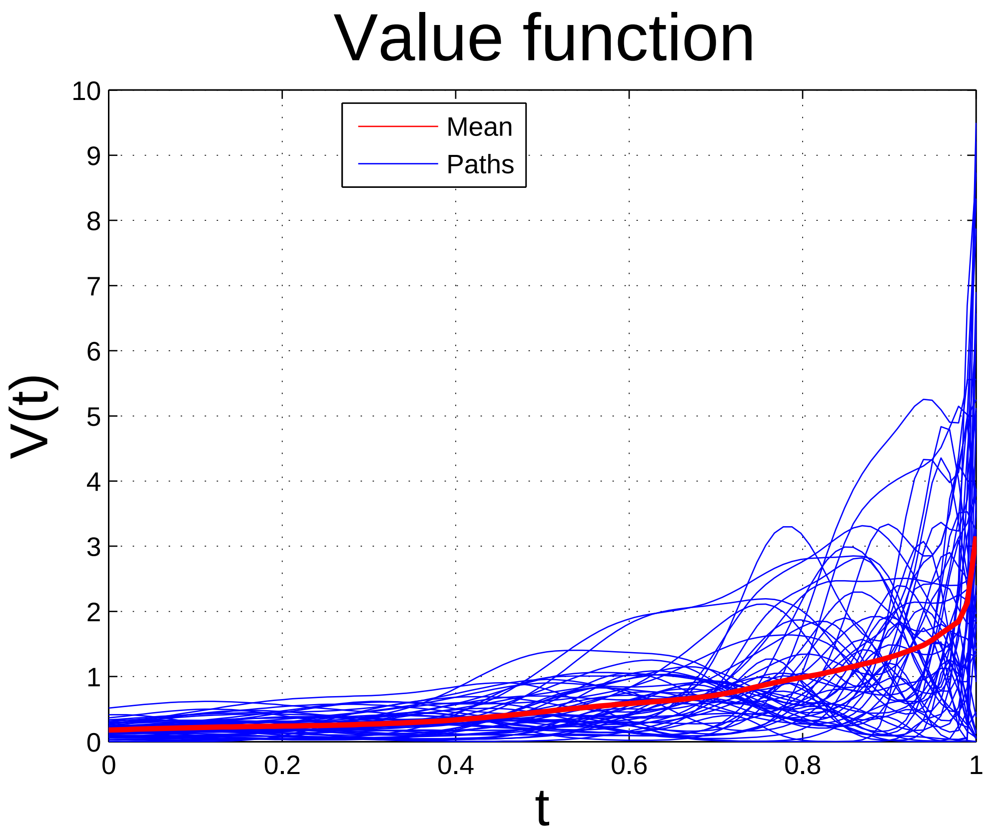

Figure 3.5:

Finally, from (3.64) we find the indifference price.

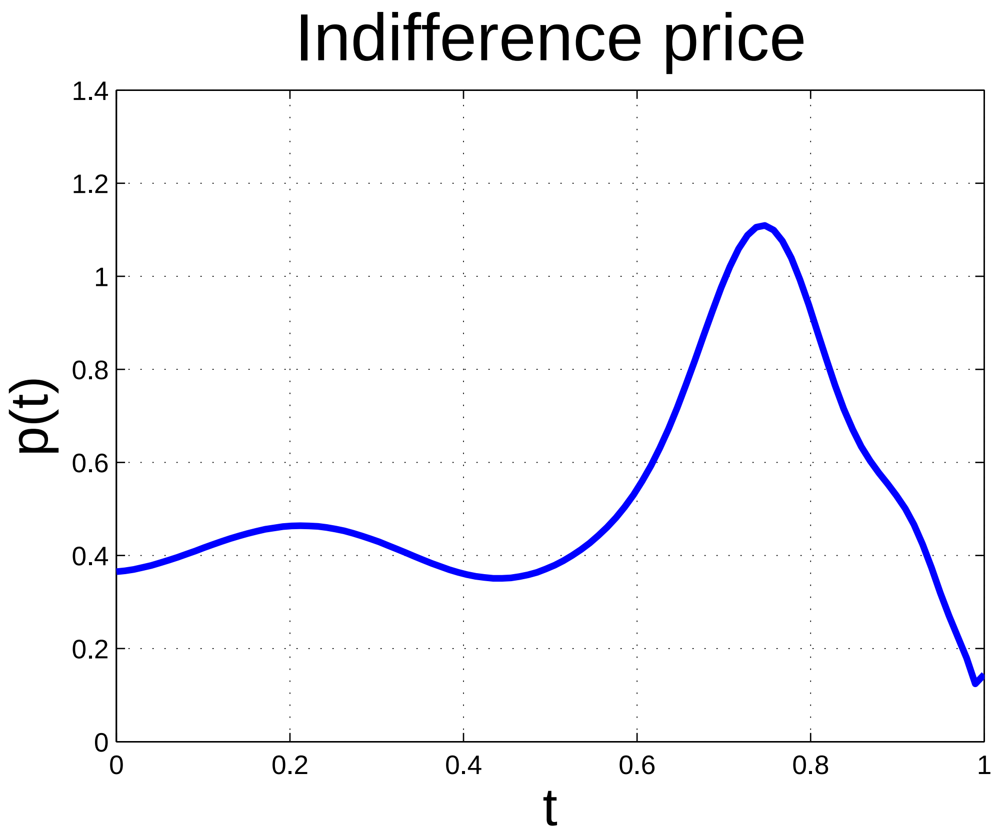

Figure 3.6:

Appendix A

Random Variables

A.1 Existence and uniqueness of the essential supremum and essential infimum of a family of random variables

The purpose of this section is to show the existence of the essential supremum and essential infimum of a family of random variables, and the almost sure uniqueness. Another purpose is to show some properties that are of interest for better understanding the Chapter 3.

Definition A.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and let $\mathcal{X}$ nonempty, a family of random variables defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$. The essential supremum $\mathcal{X}$, denoted by $\operatorname{ess\,sup} \mathcal{X}$, is a random variable X, satisfying:

- $\forall Y \in \mathcal{X}$, $Y \leq X$ a.s. and
- if Z is random variable that satisfying $Y \leq Z$ a.s. $\forall Y \in \mathcal{X}$, then $X \leq Y$ a.s.

Definition A.2. Given $\mathcal{X}$ as in definition A.1 and given $A \in \mathcal{F}$, we will say that $\pi = (H : A_1, A_2, \ldots, A_K; X_1, X_2, \ldots, X_K)$ is an $\mathcal{X}-$partition of A if:

- K is a positive integer,
- $(A_2, \ldots, A_K)$ is a disjoint partition in $\mathcal{F}$ of A and
- $(X_1, X_2, \ldots, X_K)$ are random variables in $\mathcal{X}$.

Definition A.3. For $\lambda \in (0, \infty]$, we define

$$\mu_\pi^\lambda (A) := \mathbb{E}\left[\sum_{i=1}^{K} (X_K \wedge \lambda) 1_{A_k}\right] \tag{A.1}$$

$$\mu^\lambda (A) := \sup\left\{\mu_\pi^\lambda (A) : \ \pi \text{ is a } \mathcal{X} - \text{partition of } A\right\}. \tag{A.2}$$

Note. The measure μ^λ is a non negative function defined in $\mathcal{F}$, and is finitely additive, also by the monotone convergence theorem it follows

$$\mu^\infty (A) = \sup_\pi \sup_{\lambda \in (0,\infty)} \mu_\pi^\lambda (A) = \sup_{\lambda \in (0,\infty)} \sup_\pi \mu_\pi^\lambda (A) = \sup_{\lambda \in (0,\infty)} \mu^\lambda (A) \tag{A.3}$$

Lemma A.1. *For $\lambda \in (0, \infty]$, μ^λ is countably additive.*

Proof. Consider the case $\lambda < \infty$. Let $\{A_j\}_{j=1}^\infty \in \mathcal{F}$ with $A_i \subseteq A_{i+1}, \ \forall i{=}1, 2, \ldots$ such that $A = \cup_{j=1}^\infty A_j$. Then

$$\mu^\lambda (A) = \mu^\lambda (A_j) + \mu^\lambda (A \backslash A_j) \tag{A.4}$$

$$\geq \mu^\lambda (A_j), \tag{A.5}$$

taking limits $j \to \infty$ of (A.5)

$$\mu^\lambda (A) \geq \lim_{j\to\infty} \mu^\lambda (A_j). \tag{A.6}$$

Given $\epsilon > 0$, we take j such that $\mathbb{P}(A \backslash A_j) < \epsilon$, then it follows from (A.1) and (A.2)

$$\mu^\lambda (A \backslash A_j) \leq \lambda\epsilon \tag{A.7}$$

from (A.4)

$$\mu^\lambda (A) \geq \mu^\lambda (A_j) - \epsilon\lambda, \tag{A.8}$$

by hypothesis, $\lambda < \infty$, then using (A.8)

$$\mu^\lambda (A) \geq \lim_{j\to\infty} \mu^\lambda (A_j). \tag{A.9}$$

From (A.6) and (A.9) it follows

$$\mu^\lambda (A)) = \lim_{j\to\infty} \mu^\lambda (A_j). \tag{A.10}$$

For the case in which $\lambda = \infty$. Is used (A.3)

$$\lim_{j\to\infty} \mu^{\infty}(A_j) = \sup_j \sup_{\lambda\in(0,\infty)} \mu^{\lambda}(A_j) \tag{A.11}$$
$$= \sup_{\lambda\in(0,\infty)} \sup_j \mu^{\lambda}(A_j) \tag{A.12}$$
$$= \sup_{\lambda\in(0,\infty)} \mu^{\lambda}(A) \tag{A.13}$$
$$= \mu^{\infty}(A) \tag{A.14}$$

i.e., μ^{λ} is countably additive.

□

Theorem A.1. *Let $\mathcal{X}$ nonempty family of nonnegative random variables. Then $X = ess \sup \mathcal{X}$ exists.*

Proof. By definition μ^{∞} is absolutely continuous with respect to $\mathbb{P}$. Let

$$X = \frac{d\mu^{\infty}}{d\mathbb{P}} \tag{A.15}$$

Then $\forall Y \in \mathcal{X}$ and $A \in \mathcal{F}$, we have $\mathbb{E}(1_A Y) \leq \mu^{\infty}(A) = \mathbb{E}(1_A X)$, that satisfies the first condition of the Definition A.1. If Z exists such that satisfies the second condition of the Definition A.1 then

$$\mathbb{E}(1_A X) = \mu^{\infty}(A) = \sup_{\pi} \mu^{\infty}_{\pi}(A) \leq \mathbb{E}(1_A Z) \tag{A.16}$$

We conclude that $X \leq Z$ a.s. □

A.2 Stochastic Calculus

Theorem A.2. *Tanaka's formula.*

$$|B_t| = \int_0^t sgn(B_s)\, dB_s + L_t \tag{A.17}$$

where B_t is the standard brownian motion, sgn denotes the sign function

$$sgn(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ -1 & \text{if } x < 0. \end{cases} \tag{A.18}$$

and L_t is its local time at 0 (the local time spent by B at 0 before time t) given by the L^2-limit

$$L_t = \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \left|\{ s \in [0,t] |\, B_s \in (-\epsilon, \epsilon)\}\right|. \tag{A.19}$$

Lemma A.2. *Itô's formula . Let X_t a d-dimensional semimartingale and $f \in C_2$, then*

$$f(X_t) = f(X_0) + \int_0^t \sum_{i=1}^d f_i(X_s)\, dX_s^i + \frac{1}{2}\int_0^t \sum_{i,j=1}^d f_{i,j}(X_s)\, d\left\langle X^i, X^{\hat{j}}\right\rangle_s \tag{A.20}$$

Corollary A.1. *Itô's formula for BSDE. Let X_t a d-dimensional semimartingale and $f \in C_2$ and $\xi := X_T$, then*

$$f(X_t) = f(\xi) - \int_t^T \sum_{i=1}^d f_i(X_s)\, dX_s^i - \frac{1}{2}\int_t^T \sum_{i,j=1}^d f_{i,j}(X_s)\, d\left\langle X^i, X^{\hat{j}}\right\rangle_s, \quad \forall t \in [0,T] \tag{A.21}$$

$$df(X_t) = \sum_{i=1}^d f_i(X_s)\, dX_s^i + \frac{1}{2}\int_t^T \sum_{i,j=1}^d f_{i,j}(X_s)\, d\left\langle X^i, X^{\hat{j}}\right\rangle_s, \quad \forall t \in [0,T] \tag{A.22}$$

Proof. Applying the Itô's formula to $f(\xi)$ we got

$$f(\xi) = f(X_0) + \int_0^T \sum_{i=1}^d f_i(X_s)\, dX_s^i + \frac{1}{2}\int_0^T \sum_{i,j=1}^d f_{i,j}(X_s)\, d\left\langle X^i, X^{\hat{j}}\right\rangle_s, \quad \forall t \in [0,T] \tag{A.23}$$

subtractingA.23 to A.20

$$f(X_t) = f(\xi) - \int_t^T \sum_{i=1}^d f_i(X_s)\, dX_s^i - \frac{1}{2}\int_t^T \sum_{i,j=1}^d f_{i,j}(X_s)\, d\left\langle X^i, X^{\hat{j}}\right\rangle_s, \quad \forall t \in [0,T] \tag{A.24}$$

□

Theorem A.3. *Martingale's representation. Let M a uniformly integrable martingale with $M_0 = 0$, then there exists a predectible process $Z \in H^1$ such that $M_t = \int_0^t Z_s^* dW_s$ where W is a standard brownian motion.*

Lemma A.3. *Gronwall's Lemma. Let $T > 0$, $c \geq 0$, $u(\cdot)$ a measurable non negative function in $[0,T]$, and $v(\cdot)$ an integrable non negative function in $[0,T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)ds, \quad \forall t \in [0,T] \tag{A.25}$$

then

$$u(t) \leq c\exp\left(\int_0^t v(s)ds\right), \quad \forall t \in [0,T] \tag{A.26}$$

A.3 Code

This section presents the all the developed code that was used on this work.

A.3.1 BSDE´s simulation

For the simulations we use MATLAB 7.10 R2010a.

BSDE.m

```
%% Backward Stochastic Differential Equation
% -dYt = f(t, Yt, Zt)dt - Zt dWt
% YT = xi;  eta=5; %risk aversion coefficient  beta=2;  alpha=1;  theta=alpha*beta;
% Number of iterations 70
itera=75;
% Number of partitions 120
N=120;
% Number of truncation 25
n = 25;
EsperanzaY=zeros(N);
EsperanzaZ=zeros(N);
Y=zeros(N,itera);
Z=zeros(N,itera);
pi=zeros(N,itera);
x=linspace(0,1,N);
delta=1/N;
sdelta=sqrt(1/N);
tic,
for k=1:itera
   z=zeros(N);
   y=zeros(N);
   %terminal value
   y(N,:)= Precios(0.001,N);
   %y(N,:)=0;
      for i = N-1 :-1 : 1
         for j = 1 : i
           z(i,j) = (y(i + 1,j) - y( i + 1,j + 1))./(2 * sdelta);
           y(i,j) = (y(i+1, j ) + y(i + 1, j + 1))/2 +
                    f(trunca(z(i,j), n),eta,beta,alpha) * delta;
         end
      end
```

```
   EsperanzaY=EsperanzaY+y;
   EsperanzaZ=EsperanzaZ+z;
   pi(:,k)=(z(:,1)+theta/eta)/beta;
   Y(:,k)=y(:,1); Z(:,k)=z(:,1);
   clear z;
   clear y;
end
EsperanzaY=EsperanzaY./itera; EsperanzaZ=EsperanzaZ./itera;
toc
```

Precios.m

```
function H = Precios(dt,nPeriods)
  % nPeriods  # of simulated observations
  % dt    time increment = 1
  H=zeros(nPeriods,1);
  F = drift(0, 0.1);      % Drift rate function F(t,X)
  G = diffusion(1, 3);    % Diffusion rate function G(t,X)
  SDE=sdeddo(F, G);
  SDE.StartState=0.5;
  for i=1:nPeriods
     [S,T] = SDE.simByEuler(nPeriods, 'DeltaTime', dt);
     H(i)=S(nPeriods);
  end
```

f.m

```
function
generador=f(x,eta,beta,alpha)
%generador=0.5*x^2;
%generador=abs(x);
theta=alpha*beta;
generador=x*theta+(theta^2)/(2*eta);  %-eta*((x+theta/eta)^2)/2; end
```

A.3.2 Truncated function

trunca.m

```
function H = trunca(X,n)
  H=zeros(numel(X),1);
  for i=1:numel(X)
    x=X(i);
    if x <= -(n+2)
      h = -(n+1);
    elseif x > -(n+2) && x<= -n
      h = (n^2 + 2*n*x + x*(x+4))/4;
    elseif x > -n && x<= n
      h = x;
    elseif x>n && x< (n+2)
      h = (-n^2+2*n*x-x*(x-4))/4;
    else
      h = n+1 ;
    end
    H(i)=h;
  end
```

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